

A General Theory for Positioning and Orienting 2D Polygonal or Curved Parts Using Intelligent Motion Surfaces¹

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Abstract: *In this paper we present novel algorithms to compute quasi-static and dynamic equilibrium positions and orientations (if any) of 2D polygonal or curved parts placed on general shaped 2D force field configurations. Instead of following the current limited approaches of trying to reduce the problem to a geometrical problem of symmetry, we pursued a solution to this difficult problem in a totally new and ground breaking direction. On the theoretical side, the novelty of our approach is that it puts into practice a fundamental result of piecewise-linear topology - the simplicial approximation theorem - and suggests an implementation using tools and techniques borrowed from modern computational geometry.*

1. Motivation.

In the past few years, the use of arrays of manipulators has been intensively studied as an alternative for automatic parts assembly on both macro and micro scale [1-5]. The array, with contiguous manipulators acting in unison, can be programmed to have the effect of inducing force fields on objects placed on it. The appropriate choice of force field configurations can move objects in manners that are suitable for mechanical assembly operations such as translation, rotation, positioning, orientation, alignment, sorting, parts feeder, and spatial filtering, among others [2].

A considerable amount of research effort has been devoted to the development of reliable and robust hardware implementations of the micro manipulators, however much research work is still necessary on the systems level (task-level manipulation strategies) and control of the hardware. Practical experiments confirm that arrays of manipulators induce force fields on objects placed on top of the array. The question is how to position and orient objects using these induced force fields? Is there a way to determine the quasi-static and dynamic equilibrium positions and orientations (if any) of an object placed on a general force field configuration? If so, how can we take into account friction

between the object and the array's surface and how this may affect the equilibrium positions and orientations of the object? In this paper we present an answer to these difficult questions

2. Main contribution.

Initially, we investigate the problem of determining quasi-static equilibrium positions and orientations (if any) of 2D polygonal or curved parts placed on general shaped 2D force field configurations. As will be shown in section 4, this problem can be reduced to the problem of determining the inverse image of the origin of a continuous map between the 3-dof configuration space of the 2D part and the 3D Euclidean image space of net torque and net force acting on the part. On the theoretical side, the novelty of our approach is that it puts into practice a fundamental result of piecewise-linear topology - the simplicial approximation theorem [6,7,14] - and suggests an implementation using tools and techniques borrowed from modern computational geometry [8].

Determining quasi-static equilibrium poses for a given part placed on a given force field configuration is not enough to actually generate reliable and robust manipulation strategies that can actually be used in real world applications, since the dynamic state of the part plays an important role in practice. For instance, quasi-static equilibrium analysis might indicate the part will be pushed towards a computed stable position, however, the dynamic state of the part can be such that the part will move towards a totally different direction from the one predicted by the quasi-static analysis due to its instantaneous velocity and momentum. In these cases, the part might end up stabilizing at another (different) stable position or even being thrown out of the force field region. Clearly, it is crucial to consider the dynamics of the part if we want to derive realistic, reliable and robust manipulation strategies that can be used and implemented in real world manipulation problems using vector field robot systems.

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In order to solve the dynamic equilibrium problem, we investigate an extension of the quasi-static analysis where we augment the 3D Euclidean image space of net torque and net force acting on the part, to the 6D Euclidean image space of net torque, net force, scalar velocity and angular velocity of the part. In this case, the theoretical foundation remains the same, since the dynamic equilibrium problem can still be reduced to the problem of determining the inverse image of the origin of a continuous map between the 3-dof configuration space of the 2D part and the 6D Euclidean image space. However, the implementation will require the use of computational geometry tools and techniques specific for higher dimensional spaces [9].

3. Present state of knowledge in the field and its limitations.

Current known approaches for parts manipulation using arrays of manipulators rely only on the quasi-static analysis and are derived for particular force field configurations. If we change the force field configuration, the analysis are no more valid and most of the results cannot be applied for the new configuration.

Peter Will *et. al.* proposed the use of arrays of simple micro-manipulators for assembly tasks in a DARPA/ETO supported project on cooperative MEMS robots for assembly [1-4]. The idea was to explore how to use arrays of manipulators as a miniature assembly bench. Several hardware versions were built, implemented and tested, and a physically based rigid body dynamic simulator was developed [1, 4]. To the best of our knowledge, many ideas for parts manipulation were demonstrated for the first time in rigid body dynamic simulations, such as the use of force fields for filtering, sorting and selecting a part from a specific set of parts based on the mass distribution properties of the parts being manipulated. More advanced applications involving character recognition [3], pipeline design [2] and symmetrical part assembly were also demonstrated (see <http://www.isi.edu/mass> for movies, and download of software and publications). However, no formal algorithms were suggested for manipulating a general set of parts.

Karl Bohringer *et. al.* developed what they call the “theory of programmable vector fields” [10]. This theory is an attempt to model the interaction between parts and the array of manipulators. Although theoretically correct, this theory falls short when it comes to practical real world applications due to the several unrealistic assumptions made, which prevent the algorithms generated from this theory to be robustly and reliably applied in practice. For instance, the equilibrium analysis developed in the theory does not consider friction (neither static nor dynamic) between the parts being

manipulated and the array’s surface. Therefore, the equilibrium analysis developed in the theory assumes a part will reach an equilibrium position whenever the net torque and the net force acting on the part are both zero. With friction, the net force acting on the part does not necessary need to be zero, but only smaller than the static friction force. In our view, it is clear that friction must be taken into account if one wants to develop a robust theory of programmable vector fields that can be used beyond theoretical purposes, that is, that can be used in real world applications. Karl Bohringer *et. al.* used this theory of programmable vector fields to derive planning algorithms for parts manipulation placed on a squeeze field. The use of squeeze force fields for parts manipulation was originally investigated by Peter Will *et. al.* for the case of polygonal parts, taking into account the dynamics of the part and friction [2] (for simulation movies, see <http://www.isi.edu/mass>), however no general planning algorithms were suggested. The squeeze field was chosen due to its similarities with the well-known parallel jaw gripper used in robotics. The main idea behind Karl Bohringer *et. al.* planning algorithm is to compute a squeeze function for a part placed on the squeeze field, and use this squeeze function as the input to Goldberg’s algorithm [11] for parallel jaw grippers, which automatically synthesizes a manipulation strategy to uniquely orient a part (up to symmetries).

The output of Goldberg’s algorithm is a sequence of angles that specify the directions of the squeeze field to be used. Although the planning algorithm developed by Karl Bohringer *et. al.* is a very interesting application of a well-known algorithm for parallel jaw grippers to a similar problem using squeeze fields, its main limitations are that it does not consider friction (because it is based on their theory of programmable vector fields) and it is based on a quasi-static analysis. The latter is in fact a major drawback of their planning algorithm, because the use of quasi-static analysis to derive planning algorithms for parts placed on force fields can generate erroneous manipulation strategies. This will be illustrated in the following paragraphs.

Goldberg’s algorithm initially computes the diameter function for a given part. The diameter function has local maximum and minimum points, with a local minimal point always between two local maximal points. The squeeze function is obtained from the diameter function by associating any angle between two local maximum points to the local minimum point between them. That is, when a part is squeezed by the parallel jaw gripper, it will be pushed towards the local minimum point associated with the interval angle the part finds itself by the time the parallel jaw grippers touch it. Karl Bohringer *et. al.* suggested the use of a turn function in place of the diameter function, that associates 1, -1 or 0, depending on the net torque acting on the part pushing it is clockwise, counterclockwise or zero, respectively [10]. The squeeze

function is then obtained from the turn function, in a similar fashion as the squeeze function was obtained from the diameter function.

The use of quasi-static analysis mimics the behavior of a part being squeezed by a parallel jaw gripper. However, we should bear in mind that although a squeeze field is very similar to a parallel jaw gripper, it is definitely not the same. When a part is squeezed by the gripper, the parallel jaws make a hard contact (physical contact) with the part being manipulated, whereas the squeeze field makes a soft contact. So, as soon as the gripper touches the part, there is no way the part can pass through the gripper jaws and it is guaranteed the part will remain confined within the initial interval angle between the two local maximum points and eventually it will rest at the local minimum point associated with the interval angle. This is not always the case with squeeze fields, when the dynamics of the movement is taken into account. As soon as the squeeze field acts on the part, the part can rotate and move freely because there are no “physical jaws” to prevent the part passing through one of the squeeze regions. Therefore, it is not guaranteed the part will remain confined within the initial interval angle between the two local maximum points. In fact, depending on its dynamic state, the part can have enough momentum to rotate against the force field direction and move to another interval angle associated with another local minimum point. In these cases, manipulation strategies derived from the quasi-static analysis will fail to predict the correct, real-world behavior of the part being manipulated. Therefore, it is of utmost importance to take into account the dynamic state of the part to correctly predict the behavior of the movement in a practical situation.

As an example of the importance of discriminating between “soft” and “hard” contact and the importance of taking into account the dynamic state of the parts being manipulated, consider the force field configuration of Figure 2. The temptation is to think of this force field configuration as a “glass filter funnel”. If this force field configuration was in fact a “glass filter funnel”, the “hard contact” between the parts and the filter would most probably clog it some time during the filtering process. However, in practice, this does not occur because of the “soft contact” observed along the boundary of the force field. The “soft contact” allows parts to dynamically move in and out of the squeeze regions (the boundary of the filter) due to the several dynamic interactions (collisions) between them. So, in practice, the force field configuration behaves like a “fuzzy titration” that never clogs.

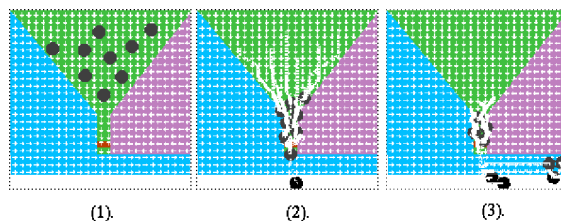


Figure 2: A force field configuration that simulates a “fuzzy titration” that never clogs [2] (see <http://www.isi.edu/mass> for simulation movies).

Lydia Kavraki suggested the use of elliptical force fields to orient and pose asymmetric parts into two stable equilibrium configurations [12], though there is no suggested way to distinguish in which of the two stable configurations the part finds itself. Peter Will *et. al.* originally demonstrated the idea of using the mass distribution properties of the parts being manipulated to perform character recognition using the relative positioning of the center of mass of the characters [2], however no general algorithms were suggested. Lydia Kavraki’s approach is also based on the mass distribution properties of the part being manipulated (its principal axis of inertia), and it can include in the analysis friction between the part being manipulated and the array’s surface, as well as tolerances in the part’s geometry. However, symmetrical parts cannot be oriented using this algorithm, and the suggested approach is specific for elliptical force fields and cannot be easily extended to other classes of force field configurations. Up to now, there is no general theory that can be applied to any force field configuration to derive manipulation strategies for any given 2D part. This is exactly our ambitious aim in this paper: the development of a theoretical foundation general enough to handle a large variety of force field configurations and parts being manipulated.

4. The theoretical foundation.

Instead of following the current approaches of trying to reduce the problem to a geometrical problem of symmetry, we propose pursuing a solution to this difficult problem in a totally new and ground breaking direction.

4.1. The simplicial approximation theorem.

The main idea is to observe that there is a continuous map $\Gamma: P \rightarrow N$ between the part’s position and orientation (the domain space P), and the net torque and net force (the image space N) acting on the part. For the frictionless case, all equilibrium positions and orientations of the part are mapped to the origin of the image space. Thus, the problem of determining all equilibrium positions and orientations of the part placed on the force field region can be viewed as the problem of determining the inverse image of the origin under the continuous map Γ . If we check for all possible positions

and orientations of the part and none of them is mapped to the origin of the image space, that is, N does not contain the origin, then we are absolutely positive there are no stable equilibrium positions and orientations of the part for the given force field configuration. So, the first step towards the solution of the problem is to check if N contains the origin of the image space. If this is not the case, then we already know there are no stable equilibrium positions and orientations. Otherwise, we have to search in P for all positions and orientations that are mapped to the origin of the image space.

Checking for all possible positions and orientations of the part that are mapped to the origin of the image space is too cumbersome a task and the problem is whether a fast check on the boundary of the domain space P can be devised. Whether a fast check exists is related to whether the map Γ and the boundary commute in the sense that

$$\Gamma(\partial P) \supseteq \partial \Gamma(P) = \partial N$$

where ∂ is the usual boundary operator of point set topology, that is, the closure minus the interior. Clearly, commutativity guarantees existence of a fast check on the boundary of P , since in this case, the image of the boundary of P contains the boundary of N (see Figure 2). So, instead of mapping all possible positions and orientations of P to obtain N and then check if N contains the origin, we would just map the positions and orientations of the boundary of P , and their images will contain the boundary of N . Having the boundary of N , we can easily check if the origin is or not inside N . On the other hand, failure of the commutativity relation requires searching inside P for all possible positions and orientations of the part that are mapped to the origin of the image space, resulting in accrued computational complexity.

In topology, there is an important class of piecewise-linear maps that “commute with the boundary” and that are called *simplicial maps*. The simplicial property refers to the behavior of a map between polyhedra, relative to triangulations of both the domain and the image polyhedra. The map should be continuous and defined over a compact, triangulable space.

Definition 1: A map $G: P \rightarrow N$ between triangulated polyhedra is simplicial if and only if every simplex of P is mapped to a simplex of N .

Corollary 1: If the map $G: P \rightarrow N$ is simplicial, then for every simplex S_n :

- $G(\mathbb{1}S_n) = \mathbb{1}G(S_n)$, if $\dim G(S_n) = n$
- $G(\mathbb{1}S_n) = G(S_n)$, otherwise

Consequently, for every simplex S_n of P , $G(\mathbb{1}S_n) \hat{E} \mathbb{1}G(S_n)$.

Proof. The first two claims are purely combinatorial results that are proved by considering all possible cases of simplicial vertex transformation. The third claim is a direct consequence of the first two. For details, see [24].

Therefore, if Γ is simplicial, there exists a fast check on the simplexes of P . In general, a continuous, even piecewise-linear, map $\Gamma: P \rightarrow N$ between triangulated polyhedra P and N fails to be simplicial. However, it can be approximated by a map that is simplicial relative to refinements of P and N .

Definition 2: A refinement of a simplex S is the decomposition of the closed simplex \bar{S} as the disjoint union of smaller (open) simplexes that form a simplicial complex.

There are many refinement rules for the simplex decomposition, such as barycentric subdivision, Q-refinement, stellar decomposition, etc. The key requirement rule is that the mesh of the triangulation resulting from iterative refinement goes to zero. Both the barycentric subdivision and the prismatic or Q-refinement satisfy this mesh requirement [14]. The following definition, lemma and theorem constitute the cornerstone of the theoretical foundation of this paper.

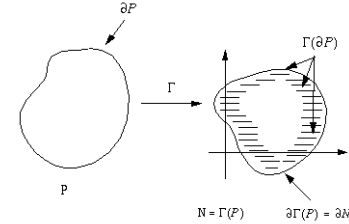


Figure 2: If $\Gamma(\partial P) \supseteq \partial \Gamma(P)$ then mapping the boundary of P is sufficient to determine whether N contains the origin of the image space.

Definition 3: A simplicial map $\tilde{\Gamma}: \tilde{P} \rightarrow \tilde{N}$ is said to be a simplicial approximation of $\Gamma: P \rightarrow N$ if

$$\Gamma(\text{star}(a^i, \tilde{P})) \subseteq \text{star}(\tilde{\Gamma}(a^i), \tilde{N}), \forall a^i \in \tilde{P}$$

where $\text{star}(a^i, \tilde{P})$ denotes the set of simplexes of \tilde{P} that have a^i as a vertex. (We shall most of the time simplify the notation to $\text{star}(a^i)$ when the underlying simplicial complex is clear from the context.)

The following lemma reveals the significance of the star relationship:

Lemma 3: The star relationship

$$\Gamma(\text{star}(a^i)) \subseteq \text{star}(\tilde{\Gamma}(a^i)), \forall a^i \in \tilde{P}$$

implies that

- $\tilde{\Gamma}$ is simplicial;

- $\tilde{\Gamma}(\mathbf{s})$ is in the closure of the carrier of $\Gamma(\mathbf{s})$;
- Γ and $\tilde{\Gamma}$ are homotopic.

Proof. The proof of the first two facts are implicitly contained in [14], Section 1.8. The third fact is a direct consequence of the second fact.

Theorem 3: Given the continuous map $\Gamma: P \rightarrow N$ between triangulated polyhedra, given a tolerance level \mathbf{e} , there exist a refinement \tilde{P} of the polyhedron P , a refinement \tilde{N} of the polyhedron N and a simplicial approximation $\tilde{\Gamma}: \tilde{P} \rightarrow \tilde{N}$ of Γ such that

$$|\Gamma(p) - \tilde{\Gamma}(p)| < \mathbf{e}, \forall p \in P$$

Proof. The construction of the simplicial approximation map is based on a piecewise linearization of the map Γ as follows. We construct a refined triangulation \tilde{N} such that for any simplex (b^0, \dots, b^n) , $n \leq 2$, of \tilde{N} , $\text{diameter}((b^0, \dots, b^n)) < \mathbf{e}$. Clearly, $\bigcup_j \text{star}(b^j)$ and $\bigcup_j \Gamma^{-1}(\text{star}(b^j))$ are open coverings of N and P , respectively. Let \mathbf{d} be the Lebesgue number of the latter open covering [14]. If we subdivide the polyhedron P fine enough to have $\text{diameter}((a^0, \dots, a^n)) < \mathbf{d}/2$ for any of its simplexes (a^0, \dots, a^n) , $n \leq \bar{n}$, then:

$$\text{diameter}(\text{star}(a^i)) < \mathbf{d}, \forall i$$

Consequently, using the Lebesgue covering theorem, for any a^i there exists a b^j such that:

$$\text{star}(a^i) \subseteq \Gamma^{-1}(\text{star}(b^j))$$

or equivalently (see Figure 3):

$$\Gamma(\text{star}(a^i)) \subseteq \text{star}(b^j)$$

There might be many b^j 's satisfying the previous condition, but if we retain one of them for each a^i , we define a vertex transformation $\Psi_0: a^i \mapsto b^j$.

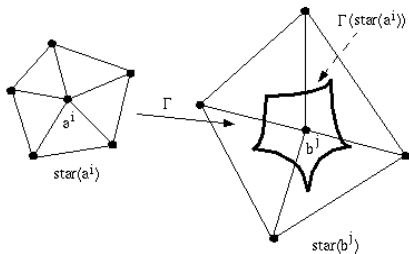


Figure 3: The polyhedron P is adequately refined whenever for any $a^i \in \tilde{P}$ there exists a $b^j \in \tilde{N}$ such that $\Gamma(\text{star}(a^i)) \subseteq \text{star}(b^j)$.

The piecewise linear extension of the vertex transformation Ψ_0 defines a topological map $\tilde{\Gamma}$. Clearly, Γ and $\tilde{\Gamma}$ are star related, so that $\tilde{\Gamma}$ is a simplicial approximation of Γ . From lemma 3, it follows that $\tilde{\Gamma}(p)$ is in the closure of the carrier of $\Gamma(p)$. From the latter, the tolerance relation is easy to show.

4.1.1. Example of application of the simplicial approximation theorem on a hypothetical map.

The following simple example serves to illustrate the utilization of the simplicial approximation theorem to compute pre-images. Let (a^0, a^1, a^2, a^3) be a tetrahedron, consider a vertex transformation $a^i \mapsto b^i$, and suppose that the convex hull of b^i is a simple quadrangle, the interior of which contains the origin O of the image space. The problem is to compute the pre-image of O for the piecewise linear extension of $a^i \mapsto b^i$. To solve this problem, triangulate the quadrangle by drawing the diagonals (b^3, b^1) and (b^0, b^2) , intersecting at z . Clearly, there exist points $x \in (a^3, a^1)$, $y \in (a^0, a^2)$ mapped to z . Draw the line (x, y) and refine the tetrahedron as shown in Figure 4. Clearly, the piecewise linear extension of $a^i \mapsto b^i$ is simplicial relative to the decomposed tetrahedron and quadrangle, since every simplex of the decomposed tetrahedron is mapped to a simplex of the quadrangle. The key point is that not much can be said about the pre-image of O without the triangulation of the tetrahedron as dictated by the simplicial approximation. With the triangulation at hand, the pre-image of O resides in (a^2, a^3, x, y) , which is the pre-image of the simplex (b^2, b^3, z) that contains O .

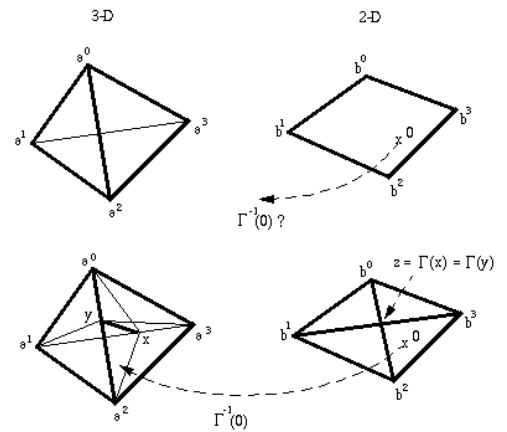


Figure 4: Utilization of the simplicial approximation theorem to compute pre-images.

4.2. Determining quasi-static equilibrium positions and orientations.

4.2.1. The frictionless case.

Consider the continuous map $\Gamma: P \rightarrow N$ between the part's position and orientation (the domain space P), and the net torque and net force (the image space N) acting on the part. For the 2D case, the part has only 3-dof ($\dim P = 3$), and the image space is embedded in the 3D Euclidean space (two real variables to define the net force on the plane, and one real variable to define the rotation around the z-axis pointing out of the plane). Note that the domain space P is the subset of the configuration space of the part that intersects the plane region defined by the force field configuration. Since the array of manipulators comprise a finite area, the plane region defined by the force field configuration is bounded (compact), and so is P .

The exact computation of P is identical to the well-known computation of the configuration obstacle space in robotics, if we imagine the force field region as the "obstacle" and the part being allowed to rotate and translate freely. In practice, we suggest to instead of going into the trouble of computing the "configuration space obstacle" to determine the exact domain region P , we can compute the Minkowski sum of the force field region and a bounding box of the part that is guaranteed to completely contain the part for all possible orientations. We can do this because all configurations of the part in which there are no intersections with the force field region are mapped to the origin of the image space, since the net force and net torque acting on the part is zero in these configurations. So, the "extra" regions added by the approximated computation of P will not influence the determination of the stable positions and orientations when the part is intersecting the force field region. Doing so, we will be able to consider both polygonal and curved parts.

For the frictionless case, all equilibrium positions and orientations of the part are mapped to the origin of the image space. The simplicial approximation theorem can be used to compute the pre-image of the origin. We propose computationally implement the simplicial approximation map $\tilde{\Gamma}$ using modern computational geometry tools [8].

The starting point of our algorithm is a coarse triangulation $h: T \rightarrow P$ where the polyhedron T has vertices a^i . By "coarse", we mean that the only requirement on this triangulation is that the number of simplexes of T be large enough to allow the existence of a homeomorphism $T \rightarrow P$. In other words, the triangulation must capture the topology of the force field configuration.

The first step is to map the vertices of the coarse triangulation of $P: b^i = \Gamma(a^i)$. Since we need a triangulation of N , the natural choice would be a triangulation of N that has $\{b^i\}$ as vertex set. There are many such triangulations. However, for

numerical reasons, it is desirable to have a triangulation optimal relative to some shape measure so as to avoid long and skinny triangles in the triangulation. This is the Delaunay triangulation problem of computational geometry [8]. A standard result of a Delaunay triangulation is that it is optimal in the sense of maximizing the smallest angle (see [8], theorem 13.11).

Generically, the Delaunay triangulation contains a unique tetrahedron (b^0, b^1, b^2, b^3) that contains the origin. If this is not the case, then we can conclude that there are no stable equilibrium positions and orientations for the part in the given force field configuration. In the usual case where the map Γ is not simplicial, the inverse images of b^0, b^1, b^2, b^3 under the coarse grid vertex transformation yield four points a^0, a^1, a^2, a^3 that could be at remote corners of P . However, if we refine P and derive an assignment rule for the new vertices, we will be able to construct a simplicial approximation map $\tilde{\Gamma}$. In fact, this assignment rule can be obtained as the vertex transformation of a simplicial approximation $\tilde{\Gamma}$ obtained by refining the polyhedron P while keeping the (Delaunay) triangulation fixed. As an approximate computational implementation of the star condition $\Gamma(\text{star}(\tilde{a}^i)) \subseteq \text{star}(\tilde{\Gamma}(\tilde{a}^i))$, we assign every new vertex of P , say \tilde{a}^i , to the vertex of N closest to $\Gamma(\tilde{a}^i)$. In other words, we have to find in what Voronoi cell $\Gamma(\tilde{a}^i)$ falls. We suggest the adoption of this approximate implementation of the star condition because it is yet another well understood computational geometry problem - the point location problem [8].

So, starting with the coarse triangulation, we continue refining P , checking whether, at the current refinement level, we have reached a simplicial map. If all edges of \tilde{P} (the refined P) are mapped simplicially under the vertex assignment Ψ_0 , that is, if for any edge (a^0, a^1) of \tilde{P} , $\Psi_0(a^0), \Psi_0(a^1)$ form either an edge or a vertex of the Delaunay triangulation, then the vertex assignment induces a simplicial map [13]. The Lebesgue covering theorem guarantees the existence of a simplicial approximation map up to a finite refinement level [6]. In other words, the refinement level at which we obtain a simplicial approximation map is guaranteed to be finite. As mentioned earlier, there are several refinement rules for the decomposition of the simplexes of \tilde{P} , and whether an isotropic (global) or anisotropic (local) refinement should be used is still an open problem.

After constructing the simplicial approximation map, we chase all simplexes of \tilde{P} simplicially mapped to the closure

of (b^0, b^1, b^2, b^3) , that is, simplicially mapped to (b^0, b^1, b^2, b^3) , (b^0, b^1, b^2) , (b^0, b^1, b^3) , (b^0, b^2, b^3) , (b^1, b^2, b^3) , (b^0, b^1) , (b^0, b^2) , (b^0, b^3) , (b^1, b^2) , (b^1, b^3) , (b^2, b^3) , b^0, b^1, b^2 or b^3 . Call Ω this collection of simplexes. Since the tetrahedron (b^0, b^1, b^2, b^3) contains the origin of the image space, each simplex of Ω (i.e. each pre-image of (b^0, b^1, b^2, b^3)) contains a region that is mapped to the origin of the image space. In other words, each simplex of Ω contains at least one stable position and orientation of the part. By theorem 3, the simplicial approximation map $\tilde{\Gamma}$ can be made arbitrarily close to the original map Γ . Therefore, given a tolerance level, we can always obtain a simplicial approximation map and use it to obtain approximations of the pre-images of the origin of the image space. These approximated stable positions and orientations of the part in the force field region can then be used to devise new algorithms and manipulation strategies to orient, position and move parts on the IMS.

4.2.2. Extending the frictionless approach to cope with friction.

In this section we will extend the frictionless method to cope with static and dynamic friction. Dynamic friction can be easily incorporated in the method during the computation of the net force and net torque acting on the part. For the static friction case, the equilibrium condition of having zero net torque and net force acting on the part, is relaxed to the condition of having the net torque and net force less than the estimated static friction force. Therefore, the existence of static friction implies that all stable positions and orientations of the part are no more solely mapped to the origin of the image space, but to a closed neighborhood of the origin. Assuming the Coulomb friction model, the module of the maximum static friction force is given by $m_s Mg$, where m_s is the static friction coefficient, M is the mass of the part being manipulated and g is the gravity acceleration. Instead of only considering the unique tetrahedron (b^0, b^1, b^2, b^3) that contains the origin of the image space, and computing its pre-images, we can extend the method described for the frictionless case by considering all tetrahedrons that intersect a sphere with radius $m_s Mg$ and center at the origin of the image space, and compute their pre-images using the simplicial approximation theorem. Call Ω this collection of simplexes. Again, each simplex of Ω contains at least one point that is mapped to the neighborhood of the origin, that is, one point corresponding to a stable position and orientation of the part when static friction is taken into account.

4.3. Determining dynamic equilibrium positions and orientations.

So far, we have presented algorithms to investigate quasi-static equilibrium positions and orientations of parts placed on a force field region. In order to investigate dynamic equilibrium positions and orientations of parts, we present an extension of the algorithm used for the quasi-static case, by augmenting the image space from 3D (net force and net torque) to 6D (net force, net torque, scalar velocity, angular velocity). Instead of just computing the net force and net torque acting on the part at each vertex of the coarse triangulation of the domain space P , we propose using numerical methods (such as Runge-Kutta) to solve the differential equations of motion at each vertex to obtain the scalar and angular velocities of the part at the current vertex. After mapping the vertices of the coarse triangulation of P , we can compute the Delaunay triangulation for the 6D image space using advanced computational geometry tools to compute Delaunay triangulations and Voronoi diagrams for higher dimensional spaces [9]. Note that we can still use the vertex transformation Ψ_0 as the assignment rule to obtain a simplicial approximation map $\tilde{\Gamma}$. After the simplicial map is obtained, we can use it to compute an approximation of the dynamic stable equilibrium positions and orientations of the part, in the same fashion we did for the quasi-static analysis.

5. Conclusion.

In this paper we investigate novel algorithms to position and orient 2D polygonal or curved parts placed on general shaped 2D force field configurations. The algorithms presented in this paper can be used to study the dynamic behavior of any particular force field configuration of interest with respect to any set of desired parts. The algorithms deal with realistic assumptions such as considering static and dynamic friction between the part being manipulated and the array's surface, and therefore are expected to provide real-world solutions. To the best of our knowledge, this is the first time an algorithm is proposed to investigate dynamic equilibrium positions and orientations of parts placed on force field regions, that is independent of the current force field configuration being analyzed.

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