General Structure of Resource Allocation Games

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Abstract

In this paper, we study a class of games that are generalizations of the minority game, and model, more generally, systems in which agents compete for a scarce resource. In particular, we study a set of games in which the demand and supply of the resource are specified independently. This allows us to study the ways in which such systems behave as the resource becomes increasingly scarce or increasingly abundant relative to demand. We find an intricate and rich structure to these games with a number of very intriguing features. Among these is the existence of a robust phase change with a coexistence region as the demand/supply ratio is varied, and the games move from scarce to abundant resources. This coexistence region exists when the amount of information used by the agents to make their choices is greater than a certain level, which is related to the point at which there is a phase transition in the standard minority game. We also discuss practical and theoretical implications of our work.

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I. Introduction

Competition for resources is ubiquitous in social and biological systems. Animals foraging for food, companies competing for market share, clients competing for bandwidth on the Internet and politicians competing for votes are just a few examples. Although each specific system has its own particularities and constraints, we believe that there are some general underlying dynamics that are at the heart of the problem of competition for resources, and that drive the resulting endogenous and dynamical pattern of allocation of those resources.

To help us understand these dynamics, it is useful to construct very simple models which capture the essence of one or more of what may be the basic and underlying dynamics that play a role, generically, in systems of competition for resources. One such dynamic is associated with being innovative, being different, or being in the minority. It is very clear that some dynamic such as this is important in a number of different systems of competition for resources. Animals that find a rarely used pasture may be rewarded with plentiful amounts of food. Drivers commuting to work along rarely used roads may arrive at their destinations more quickly and less drained than others who travel overcrowded freeways. On the other hand, minority seeking is not the only dynamic at play. Political power is often associated with larger groups. In financial markets one may profit, at least for a time, by pursuing a strategy consistent with what most others are doing—the trend is your friend.

In any case, it is clear that minority dynamics plays an important, if not the only role in competition for resources. One model for minority seeking that is particularly interesting is the by now celebrated minority game.[1,2,3] In this game, an odd number, N, of agents play a repeated game at each time step of which each agent joins one of two groups. Agents are awarded a point if they are in the smaller (minority) of the two groups, and are awarded nothing if they are in the larger group. Decisions are made by the agents using strategies particular to each agent, and agents' strategies use a set of publicly available information as input. (See below for more details.)

The minority game can also be thought of as a resource allocation game in which N agents compete for resources from one of two suppliers (the two groups). Each supplier has available at each time step (N-1)/2 units of the resource, and each agent must decide, at each time step of the game, from which supplier to request one unit of the resource. In the simplest version of the game described above, if a supplier is overloaded so that more resource is requested than he has available, no requesting agent gets any resource. If the supplier is underloaded, then the agents requesting a unit of resource can be satisfied. When the minority game is cast in this way, it becomes clear that a very natural extension of the game is to

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[1] D. Challet and Y.-C. Zhang, *Physica A*, **246**, 407 (1997).

[2] R. Savit, R. Manuca and R. Riolo, Phys. Rev. Lett. **82**, 2203 (1999); R. Manuca, Y. Li, R. Riolo and R. Savit, *The Structure of Adaptive Competition in Minority Games***,** *Physica A* ***282**, 559 (2000).*

[3] For references to the extensive literature on this game, see the excellent web site http://www.unifr.ch/econophysics/minority.

decouple the supply of resource available from the number of agents. So we now consider a set of games in which N agents (not necessarily odd) compete for resources from G suppliers, each of whom has available, at each time step of the game, C/G units of resource. (Except in the Appendix, we constrain C to be an integer multiple of G.) The description of the game is completed by specifying the way in which agents make their decisions, and the payoff that agents receive, given the capacity of a supplier and the number of agents requesting a unit of resource. The special case of odd N, G=2 and C=N-1 along with a simple binary payoff to the agents is the standard minority game.

In this paper, we shall show that the set of generalized resource allocation games, of which the minority game is a special case, has a very rich and intriguing structure. In particular, we are able to identify a number of regions with qualitatively different behaviors as a function of the quantity of available resource, C, relative to the demand, N. As one moves away from the special case of the minority game, say, by varying C with fixed N, one encounters a phase change as a function of C. For increasing C, this phase change marks a transition from a region of limited resources to one in which the resource is abundant, while for decreasing C a similar phase change occurs as we move from a region of limited resources to one in which the resource is scarce. Moreover, these phase changes are associated with interesting coexistence regions. Within each of these regions (abundant, limited, and scarce resources) there is further fine structure. In addition, all of this structure in the supply/demand variables (N and C) is modulated in an important way by the amount of information that the agents use to make their choices. Finally, although we will not discuss this in detail in the present paper, we have discovered some interesting scaling relations in these games.[4]

It is important to note that the set of resource allocation games we will consider share with the basic minority game the general structure that innovation, or being different in a very general sense, is rewarded. In particular, these games do not reward coalition formation *per se* or related dynamics that are often associated with resources such as political power. The relationship between such majority seeking or aggregating dynamics and the general innovation dynamics captured in the games discussed here is most interesting and deep, but is well beyond the scope of the present paper.

The rest of this paper is organized as follows: In the next section we describe the models of resource allocation we have studied. In section III we present the main results of our simulations as well as some analytic results. Section IV consists of a summary and discussion of the main features of our results, as well as a discussion of their theoretical and practical implications and suggestions for further research. A theorem that relates results in a set of games with abundant and scarce resources, respectively, is proven in Appendix A.

II. The Basic Resource Allocation Game

Consider a game in which N agents compete for a resource from one of G suppliers. In the games discussed here,

[4] Y. Li, S. Brueckner, H.V.D. Parunak, J. Sauter and R. Savit, in preparation.

G=2. We will consider the games with more than two suppliers elsewhere. At each time step of the game, each supplier has available C/2 units of the resource. At each time step of the game, each agent chooses one of the two suppliers as a source for one unit of the resource. The agents make their choices of which supplier to choose at a given time step, using a mechanism similar to that used in the minority game. In particular, we endow each agent with s (in the games considered here, s=2) strategies. Each strategy is a look-up table in which data from a set of publicly available information is used as input to determine an agent's decision. Specifically, the set of publicly available information for the agents is the historical time series of which of the suppliers was overloaded as a function of time, and a strategy of memory m uses that information for the last m time steps. A supplier is underloaded at a given time step if the number of agents requesting a resource is less than or equal to C/2, and is overloaded otherwise. We indicate underloading of a supplier by +, and overloading by -. Then, in general, the state of the system at any given time is defined by a two-tuple (a,b), where a indicates the state (over- or under-loaded) of supplier 0 and b indicates the state of supplier 1. Although there are, in principle, four possible states, the number of accessible states depends on the relative values of N and C. If $N \le C$, then states (+,+), (+,-) and (-,+) are possible. If N > C + 1, states (-,-), (+,-) and (-,+) are possible. For the special case N = C + 1, this game reduces to the minority game and only states (+,-) and (-,+) are possible. Thus, if the agents use information in their strategies from the last m time steps of the game, the dimension of the strategy space will be 3^m, except if N=C+1, in which case the dimension of the strategy space will be 2^m. An example of an m=2 strategy for the case N>C+1 is shown in Fig. 1.

m=2 history of suppliers	Supplier choice at current time
(-,-) ; (-,-)	0
(-,+) ; (-,-)	0
(+,-) ; (-,-)	1
(-,-) ; (-,+)	0
(-,+) ; (-,+)	1
(+,-) ; (-,+)	0
(-,-) ; (+,-)	1
(-,+) ; (+,-)	1
(+,-) ; (+,-)	0

Figure 1. An example of a strategy table for m=2 in the case in which N>C+1.

An agent must choose which of its two strategies to play at a given time. An agent will choose to play the strategy that would, up to that point in the game, have been responsible for the greatest gain for the agent, had that strategy been played for all past times of the game. Thus, the relative ranking of an agent's strategies will depend on the payoff to the agents. In this paper, we examine games with two different payoff schemes. The first, called binary payoff, awards one point to each agent using an underloaded supplier, while agents using an overloaded supplier get nothing. These same awards are made to strategies to determine their relative rankings. The second payoff scheme, called partial satisfaction, awards one point to each agent using an underloaded supplier, while each agent using an overloaded supplier is awarded a fraction of a point equal to C/(2n), where n (>C/2) is the total number of agents using that supplier. The same scheme is used to award agents' strategies. Specifically, the strategy of an agent that is actually played is awarded the same points as the agent. To evaluate a strategy not played, one awards one point to that strategy if it would have chosen an underloaded supplier at some time step, and awards no points (binary payoff) or C/(2n) points (partial satisfaction) if it would have chosen an overloaded supplier, where n is the number of agents actually using the overloaded supplier at that time step. Note that these awards are made assuming the same distribution of agents among the suppliers as actually occurred. No correction is made for the fact that had the unplayed strategy been played, the distribution of agents might be different. Thus, this strategy ranking scheme is similar to that of the naïve agents used in the first studies of the minority game. Most of the results reported in this paper are for games played with binary satisfaction. We will also discuss the most salient differences encountered when the payoff scheme is that of partial satisfaction.

We have not studied the resource allocation game with sophisticated (or, more awkwardly, non-naïve) agents, in which strategy rankings take account of what would have been the altered distribution of agents distributed between the two suppliers. We expect that using such sophisticated agents will induce changes in our results that parallel those changes that occur in the minority game. Finally, we have also studied another reward scheme related to partial satisfaction, called probabilistic satisfaction, in which, for those agents using an overloaded supplier, C/2 are randomly chosen and awarded one point, while the rest get nothing. In this version, a sub-set of strategies pointing to an overloaded supplier are also randomly chosen and awarded a point. While many of the results are the same for partial and probabilistic satisfaction, there are some interesting differences. This will be discussed further elsewhere.

One important reason for considering a variety of different payoff schemes, is that different schemes may be relevant to different natural or engineered systems. Consider, for example, the problem of electricity supply. The binary payoff scheme is relevant to those situations in which an overload of demand trips a circuit breaker, completely shutting off supply from the oversubscribed supplier. On the other hand, partial satisfaction is more relevant to a situation in which an overload of demand results in brown-outs.

Metrics.

Several different metrics are of interest for this game. First, and perhaps most importantly, we consider σ^2/N, where σ is the standard deviation of the number of agents requesting resources from one of the suppliers over a run. In the minority game, this quantity is inversely related to the typical number of agents in the *minority* group, and thus is an inverse measure of the extent to which the system's limited resources are being used. I.e., when σ is small, the size of the minority group will typically be close to 50% of the agents. If $N \neq C+1$, then σ is not as simply related to resource utilization. However, it is still an aggregate measure of system uncertainty and in that sense may be related, in more realistic scenarios, to the computational effort that an agent must expend to make decisions about which supplier to patronize. Finally, in a statistical mechanical treatment of the minority game, σ^2/N is closely related to a Hamiltonian. Although we have not explicitly carried out a statistical mechanical treatment of the more general resource allocation game, we expect that σ^2/N will play a similar role there. We will also report this quantity averaged over a number of different runs with the same parameter settings (i.e., the same values of C, N and m). That is, σ^2/N is computed for each run with a given parameter setting, and that quantity is averaged over different runs with the same parameters. This is the same metric that has been used extensively in studies of the minority game. Runs with the same parameter settings differ in the particular (randomly generated) strategies that are assigned to the agents at the beginning of the game. Different runs also differ, of course, in the particular string of random numbers that are used to determine the outcome of specific probabilistic events.

We will also be interested in observing the spread of values of σ^2/N among different runs with the same values of C, N and m. So, we shall study $\eta \equiv \sigma[\sigma^2/N]$, the standard deviation of σ^2/N over runs with the same C, N and m.

Although it may not be as theoretically fundamental, the average payoff to the agents, and the standard deviation of that payoff is significant. So, we shall consider W, the average award accumulated per time step by each agent in a given run. We will also consider $\sigma[W]$, the standard deviation of that measure over different runs with the same parameter settings. We can construct some illuminating stories concerning the outcome of games for various parameter settings by considering these quantities. This will be discussed elsewhere. Also, as we shall discuss elsewhere, a consideration of W suggests a possible relationship between our work and satisfiability problems.

III. Results

A. General Landscape of the Resource Allocation Game

We have computed the basic metrics described in the last paragraph for a range of values of N, C and m and for different payoff schemes. The basic results are illustrated in a sequence of three-dimensional plots shown in Figs. 2-5. In these figures, we have used the binary payoff scheme. (Differences with partial satisfaction will be discussed below.) 13 different runs of 10,000 time steps were performed for each value of N, C and m. Figure 2 shows a sequence of plots of σ^2/N as a function of N and C for fixed m. Figure 3 shows η for the same range of parameters. In Fig. 4 we plot W, and in Fig. 5 we plot $\sigma[W]$, again for the same range of parameters.

We shall have much more to say about these plots below, but for the moment, note the following gross features:

Fig. 2: σ^2/N as a function of N and C has seven distinct regions. At the extremities, (large N, small C and large C small N) are two regions in which σ^2/N is fairly smooth as a function of N and C. As we move into toward the diagonal, we pass into areas in which the dependence of σ^2/N on C and N is rougher. Moving further toward the diagonal from either direction, σ^2/N decreases and, at least for m>2 has a smoother dependence on C and N.

Finally, very close to the diagonal σ^2/N increases, reaching its maximum value near C=N. This general structure obtains for all values of m in these graphs. However, there are differences. Note, in particular, that the distinctions among the regions are most clear and the central peak most pronounced for m=3. This is significant since the critical value of m, m_c, at which there is a phase transition for the standard minority game (N=C+1), is close to 3 for the range of N shown in these graphs.

In Fig. 3 we plot $\eta=\sigma[\sigma^2/N]$. In the context of the standard minority game, this quantity has a decidedly different behavior across the phase transition, being large in the low m (symmetric) phase, and small in the high m (broken symmetry) phase. In these figures, we see, generally, that η is small in the regions where σ^2/N varies smoothly, and is large in the regions where σ^2/N is rough. However, we also see that, for larger m, η is large in a well-defined region on either side of the central peak, corresponding to the region in which σ^2/N decreases rapidly. We shall say more about these observations below.

In Fig. 4 we plot the average award, W, received by the agents as a function of N and C, for different values of m. The general structure here is not surprising. First, we note that for N<<C all agents are satisfied at all time steps of the game, while for N>>C, no agent, typically, is satisfied. The interesting region is in a range surrounding N≅C, which marks the transition from complete to no satisfaction. Here we see that for m near 3 (which is close to m_c for this range of N and C), the transition region is most narrow and well defined.

Finally, in Fig. 5 we plot $\sigma[W]$. Here we see a general structure that is reminiscent of Fig. 2, in that there are, typically, seven regions for most values of m as a function of N and C. However, there is an additional dip near the center of the large peak, close to the region N≅C.

It is also worth looking a little more closely at the behavior of σ^2/N near the central peak in these graphs. To that end, we plot in Fig. 6, σ^2/N as a function of N for fixed C, and for different values of m. Note that, for this range of N and C, σ^2/N is much larger for the minority game configuration, N=C+1, than for neighboring values of N for $m \leq 3$, while for m>3 the value of σ^2/N dips at the minority game configuration. Thus, as a function of the external control variables, N, C and m, the much-studied minority game is something of a singular point in the space of resource allocation games. We will address this further below.

The careful observer will not have failed to note the apparent symmetry, particularly in Figs. 2 and 3, on either side of the region near the minority game configuration. In fact, it is possible to prove that the time series of group sizes, and therefore σ and η, are identical in the two binary satisfaction games played with a given N and $C=N-1\pm d$ for any given d. In particular, given an initial set of strategies to N agents, the group choices made by those agents are identical in the two games played with those agents and with C=N-1-d and C=N-1+d. We will refer to this result in our ensuing discussions. A proof is presented in an Appendix. The precise result relies on an assumption that all ties between strategies are broken in the same way (e.g. using the same random number generator with the same seed), and on binary satisfaction. In the case of partial satisfaction, although this result is not correct in detail, the symmetry d → -d is empirically valid in an average sense for d not too large.[5]

B. The Nature of the Regions (Binary Satisfaction)

In this subsection, we first describe the general nature of the seven regions delineated in the previous subsection. Sub-subsection v. discusses a qualitative phase diagram, which summarizes the most salient features of our study.

i. Far away from the diagonal

First and simplest, consider large N and small C. If N is much larger than C, then, with extremely rare exceptions, the result of the game at every time step will be (-,-), that is, both suppliers will be overloaded. This means that only one entry in each strategy will be sampled. Moreover, since either group will (almost) always be overloaded, strategies will (nearly) always be tied, since no strategy will ever be rewarded. Of the N agents, roughly N/4 agents will have both strategies responding with group 1 to the input of a string of m (-,-)'s, N/4 agents will have both strategies responding with group 0 to the input of a string of m (-,-)'s, and N/2 agents will have one strategy responding 1 and one strategy responding 0. Ties between the strategies are determined by a coin flip. Therefore, we expect that in this very high N very small C regime, σ will have a value associated with a random binary process of N/2 instances, so that $\sigma^2/N \sim 1/8$.[6] This is consistent with the fairly smooth large N, small C region in the graphs of Fig. 2.

It is not difficult to see that a similar argument should obtain in the very large C, very small N regime. Here, with extremely rare exceptions, the result of the game at every time step will be (+,+), and reasoning similar to that used in the previous paragraph will apply. Under the condition that each supplier will always be underloaded, both an agent's strategies will always be tied. So, we expect to see behavior similar to that observed for large N, small C in the region of small N, large C. Indeed, that is what we observe in Fig. 2.

ii. A little closer in

As we move toward the diagonal from large N, small C or small N large C in Figs. 2, we encounter regions in which σ^2/N is generally larger and more variable. At this transition, there is also a pronounced qualitative change in η (Figs. 3), at which η goes from a region with a very low value nearly independent of N and C to a region of much higher and more variable values. Although it is difficult to see in Figs. 2 and 3, more extensive simulations show that as m increases, the boundary between the smooth and rough regions

[5] A modified version of partial satisfaction, in which more than one point is awarded to agents that subscribe to an underloaded supplier, restores the d → -d symmetry, empirically, for all d.

[6] Actually, the average value of σ^2/N is somewhat smaller than 1/8 in these figures. The reason is that in the beginning of the game, a random sequence of results for m time steps is generated to set the initial scores of the strategies. This initialization breaks the ties of some pairs of strategies that would otherwise have contributed to σ by virtue of having two different responses to the sequence of all (-,-). In fact, the way in which this is accomplished in the binary satisfaction game is very interesting and will be discussed below.

moves out in N for fixed C. This transition occurs at values of N and C for which there is some reasonable probability for a result other than (-,-) to occur. When this happens, different elements of the strategy tables are sampled, and the simple argument presented for the outer two regions no longer applies. In particular, consider values of N and C close to the boundary between the rough and smooth regions of σ^2/N. For most of the time steps the outcome (-,-) will obtain, but occasionally a different outcome will occur. It is very interesting to look in somewhat more detail at this transition. To that end, look at Fig. 7, in which we plot σ^2/N as a function of N for C=600 and m=10. (We have chosen such large values of C and N in order to see clearly the fine structure in the system. Other, smaller values of C and N show similar behavior.) In this plot we show the outcome of 32 individual runs for each value of N. The solid line shows the mean of these values. Note that for very large N, nearly all runs have σ^2/N near 1/8, as we expect. For somewhat smaller values of N, though, there appear to be two and possibly three other branches, coexisting with the $\sigma^2/N\sim1/8$ branch. The most prominent of these is a branch that begins near zero for small values of N and rises to $\sigma^2/N\sim1/16$, where it asymptotes. That branch disappears for large enough N. In addition to these branches, there are some outliers, runs with much larger values of σ^2/N. To understand the $\sigma^2/N\sim1/16$ branch, consider the following: As explained above, when N>>C, the system will always be in the (-,-) state leading to $\sigma^2/N\sim1/8$. However, when N is large but not much larger than C, there will be some significant probability that the system will be in either a (+,-) or a (-,+) state at some time step. Suppose that N is large, so that the system is almost always in a (-,-) state, but that at some time step, the system is in a (+,-) state. Until (+,-) occurs, both suppliers are always overloaded, so all strategies, regardless of their response to the string of m (-,-)'s, will be tied. In particular, all the roughly N/2 agents whose two strategies respond differently to the string of m (-,-)'s will have their two strategies tied, and breaking those ties with coin flips leads to the result $\sigma^2/N\sim1/8$. However, it is not difficult to see that the advent of a state different from (-,-), say (+,-) will result in breaking the tied rankings of about half of those N/2 strategy pairs. If the occurrence of a state different than (-,-) is rare, then the main effect of that occurrence will be to break those ties, and so the effective number of agents that contribute most of the time to σ will be 1/2*N/2=N/4. This leads to a value of $\sigma^2/N\sim1/16$. Note that this branch disappears for large enough N, since for N large enough it becomes nearly impossible for the system to be in any state other than (-,-). Note also in Fig. 7, the existence of two other branches, one at $\sigma^2/N\sim3/16$ and one at $\sigma^2/N\sim1/8$ These values of σ^2/N can be explained if the effective number of agents that contribute by random choice to σ is 3N/4 and N/2, respectively. We believe that an argument similar to the one used to explain the $\sigma^2/N\sim1/16$ branch can be used here also. The detailed construction of that argument is left as an exercise to the reader. Finally, we see in Fig. 7, that there are a number of runs with much larger values of σ^2/N. This cloud of outliers occurs, roughly, for values of N for which the branches of σ^2/N different from 1/8 occur. This suggests that the outliers are due to the relatively rare occurrence of states different from (-,-). Indeed, this is the case. In Fig. 8 we present a time series of n_1, the occupation of group 1 for one of the runs with a large value of σ^2/N. We see in this time series alternations of epochs during which n_1 is greater than N/2, with epochs during which n_1 is less than N/2. Changes in the magnitude of n_1 relative to N/2 are always preceded by the system's being in a state different from (-,-), either (+,-) or (-,+). The conclusion of this observation is the following: Both the branches in which $\sigma^2/N \sim 1/8$, 3/16, or 1/4 as well as the outliers with very large values of σ^2/N owe their existence to the rare but non-negligible occurrence of states different than (-,-). For some runs with some initial distribution of strategies to the agents, the effect of these states is to drive the system to relative stability in which the occupancies of the groups generally hover near N/2 with some σ (the value of which may have been conditioned by the rare occurrence of a (+,-) or (-,+) state). However, for other runs, the occurrence of these states destabilizes the system for a period of time, resulting in the bursting phenomenon seen in Fig. 8. It is unclear whether the initial distribution of strategies determines whether a particular run will be an outlier or will be in one of the branches of σ^2/N, and if so, which one. It may also be that the answer to this question is sensitive to the initial m-history of publicly available information, or to the particular way in which ties between strategies are broken. In any case, we have not examined this closely. (See, however, the discussion of a similar phenomenon described below.[7])

iii. Regions near the diagonal—the central peak

As we continue to approach the diagonal, we see, in Figs. 2, that σ^2/N decreases and the landscape becomes less rough, culminating in a smooth valley. (Fig. 7 offers a more detailed view.) To understand this region it is most useful to reverse direction and first discuss the central peak in σ^2/N in Figs. 2. First, as noted above, with respect to the control variables, N, C and m, the minority game configuration, N=C+1 is a special point in the space of resource allocation games. σ^2/N for that configuration is either much higher or much lower than neighboring values. The reason for this singular behavior is related to the special nature of the strategy space for the minority group configuration. As mentioned in section II, for values of $N \neq C+1$, there are three possible states for each time step. If N>C+1 these states are (+,-), (-,+), and (-,-) while if N<C+1 possible states are (+,-), (-,+), and (+,+). But if N=C+1 only states (+,-) and (-,+) are possible. For the game with memory m, therefore, the dimension of the accessible strategy space is 3^m if $N \neq C+1$, but only 2^m if N=C+1. From many studies of the minority game, we know that σ^2/N varies considerably with changes in the ratio of the dimension of the strategy space to N (in the standard minority game this is $z=2^m/N$). Therefore, it is not surprising that a large change in the dimension of the strategy space should

[7] As we shall explain below, one reason we have not pursued this further, is that, unlike some of our other results, some of this fine structure is dependent on the assumption of binary satisfaction, and disappears when the payoff scheme is changed to partial satisfaction. We have, however, examined a similar issue concerning a more robust feature of the games, discussed in Section IV, below.

result in a large change in σ^2/N for a resource allocation game. In fact, as we shall show in a forthcoming publication, the values of σ^2/N do not vary abruptly from the minority game configuration to other nearby resource allocation game configurations when the results are plotted with respect to dynamically relevant scaling variables.[4] For the moment, however, we can understand the general behavior of σ^2/N near the minority game configuration in the following way. First, in Fig. 9, we plot σ^2/N as a function of m for C=N+1 (the minority game configuration), and for C=N and C=N+2. (By the theorem proved in the Appendix, the results for C=N and C=N+2 are identical. The differences between the curves for these two values of C are the result of stochastic variation.) The curves for C=N and C=N+2 have similar shape to the curve for the minority game, but their minima (denoted by m_c for the minority game, and by m_c* for games with capacity one away from the minority game) are offset in m. Both the similarity of these curves and the offset of the minima can be understood if we suppose that the important scaling variable in resources allocation games is D/N, where D is the dimension of the strategy space accessible to the system. (This is correct if all possible input signals occur with equal probability).[4] And, indeed, if one plots σ^2/N as a function $z=2^m/N$ for N=C+1 and σ^2/N as a function of $z'=3^m/N$ for N=C or N=C+2 the plots are very similar.[8] Now, as we see in Fig. 9, for the range of N plotted in these graphs (i.e. N near 30), the critical value of m for the minority game configuration is close to 3 (i.e. at $z \approx 1/3$). Consequently, for $m \leq 3$, σ^2/N is larger for N=C+1, than for neighboring values of N, while for m>3, σ^2/N is smaller for N=C+1 than for neighboring values of N.

[8] Similar, but not necessarily identical. See Ref. 4.

iv. Regions near the diagonal—the valley

Having explained the abrupt change of σ^2/N from the minority game configuration to neighboring configurations, we continue our discussion to study the system somewhat further away from the diagonal, N=C+1. To do so, consider increasing N for fixed C. We see that σ^2/N decreases fairly smoothly with N, reaching a minimum, and then beginning to increase. It is instructive to look again at Fig. 3. Notice that η is small very close to the diagonal (N=C+1), and, for sufficiently large m, gets large in the region where N is rapidly decreasing.

To help understand what is happening in this region, look at Fig. 10 in which we plot σ^2/N as a function of N for C=200 and various m for 32 different runs for each value of N. (Qualitatively similar graphs obtain for smaller C and correspondingly smaller N, but using larger C and N here allows us to see effects more clearly.) Note, first of all, that σ^2/N for the minority game configuration is considerably different than for neighboring values of N, as we saw earlier. (Here, however, the cross-over from markedly larger values of σ^2/N to markedly smaller values for N=C+1 relative to neighboring configurations occurs at m=6, which is near the critical value of m in the standard minority game for N=201, as we expect.) Next, note that there is a region, most easily seen in the graphs for $m \geq 5$, in which the values of σ^2/N for different runs separate into two quite distinct groups. These correspond to the regions in which the average value of σ^2/N is rapidly decreasing in the Figs. 2, and in which η is large in Figs. 3. This is a remarkable result and shows that for a given C and a range of N, there is a well-defined coexistence region between two very distinct phases in these resource allocation games for large enough m.[9] For very small m, the coexistence region disappears, being replaced by a relatively smooth transition in σ^2/N as a function of N, in which all runs fall, very roughly, along a single (diffuse) curve. To emphasize the importance of this coexistence region, we anticipate a result to be discussed below, and comment that this coexistence region, with two markedly different values for σ^2/N, persists even if the payoff function is changed to partial satisfaction. Unlike some of the fine structure discussed in Section III.B.ii, this transition appears to be a robust feature of these games. Finally, as N increases past the coexistence region, we see the emergence of the behaviors described in section III.B.ii, above.

How do we understand this coexistence region? First, note that the values of σ^2/N for both branches in the coexistence region are less than the random choice game (RCG) value of 0.25. So even in the higher branch there is coordination among the agents' choices of suppliers. Indeed, the values of C, N and m in Figs. 10 correspond to the region of the curves in Fig. 9 somewhat to the right of the minimum. If we look at samples of the time series from runs in the upper and lower branches, we see, not surprisingly, that runs from the upper branch show sizable fluctuations in group size about the mean, and those from the lower branch show much smaller fluctuations. However, runs from the lower branch oscillate (with their small fluctuations) about a population size of N/2 in each group. Thus, for almost all time steps of the game, the system returns a result (-,-). On the other hand, runs from the upper branch, with their larger oscillations, often return a system result of (+,-) or (-,+). What this means is the following: Recall that for the minority game configuration, only results (+,-) and (-,+) are allowed, because of the constrained relation between N and C. Resource allocation games with configurations of C and N that are close to the minority game will dynamically generate results that put the game, much of the time, in the same strategy subspace to which the minority game is confined by construction. Consequently, these nearby configurations have results that are qualitatively similar to those of the minority game. (As alluded to earlier, the fact that the minority game itself has results that are *quantitatively* distinct from nearby configurations is a function of the fact that the results are not plotted with respect to the most dynamically meaningful variables. [4]) As N increases away from the minority game configuration the system crosses over to a region in which the system's behavior is dominated by a single outcome, either (+,+) or (-,-), depending on whether N is large or small relative to C. Thus, the central peak area in Figs. 2, including the minority game configuration, is a region of competition for limited

[9] We consider here the behavior of σ^2/N as a function of N for fixed C and m showing that there is a coexistence region for large enough m. That coexistence region is centered near the minimum of σ^2/N as a function of N Later we will show that the position of that minimum (as a function of N for fixed C), varies slowly with m.

resources. Away from the central peak, we move to regions in which resources are either abundant or scarce.

Of course, the remarkable thing is the existence of an abrupt change, for large enough m, in the general structure of the game as N varies for fixed C, and a bifurcated coexistence region rather than a smooth crossover. Also intriguing is the disappearance of that coexistence region and the establishment of a smooth crossover for small m.

The qualitative change in the structure of the coexistence region with m raises the question of whether there is something special about that value of m below which coexistence vanishes. Indeed, we have found that the value at which coexistence vanishes is close to m_c^*, the value at which there is a phase transition in σ^2/N as a function of m, *away* from the configuration N=C+1. See subsection v, below for a further discussion of this important point.

Although it is nontrivial that the transition from the limited resource phase to the scarce or abundant phase proceeds through a bifurcated coexistence region, it is possible to estimate roughly where that transition region should be. The general position can be estimated by the following argument. Consider a collection of independent agents playing a random choice game, so that at each time step of the game, each agent chooses independently and with equal probability to join one of the two groups. Suppose further that N>C. Then, for a given N and C, it is possible to compute the probability $P(n_0)$ for any group sizes (n_0 and n_1=N-n_0) for the two groups. We can segregate those outcomes into classes corresponding to whether the outcome is (-,-) (class I) or is (+,-) or (-,+) (class II). The probability to have n_0 agents in one group and n_1=N-n_0 agents in the other group is simply

$$P(n_0) = \frac{N!}{n_0!(N-n_0)!}\left(\frac{1}{2}\right)^N \qquad (1)$$

so that the probability to have the outcome (-,-) (class I) is

$$P_I = \sum_{n_0=C/2+1}^{N-C/2-1} \frac{N!}{n_0!(N-n_0)!}\left(\frac{1}{2}\right)^N \qquad (2)$$

while the probability for an outcome in class II is

$$P_{II} = 2\sum_{n_0=0}^{C/2} \frac{N!}{n_0!(N-n_0)!}\left(\frac{1}{2}\right)^N . \qquad (3)$$

We take as a zeroth order estimate for the position of the transition region as a function of C for fixed N, that value of C for which $P_I = P_{II}$.

For large N and C, P_I and P_{II} become integrals over Gaussians,

$$P_I \approx \int_{(\lambda-1)/2}^{(1-\lambda)/2} dx e^{-Bx^2} \; and$$
$$P_{II} \approx 2\int_{-1/2}^{(\lambda-1)/2} dx e^{-Bx^2} \qquad (4)$$

where λ=C/N, B is a constant proportional to N, and x=$\frac{2n_0-1}{2N}$.. In approximating (3) by (4), the usual qualifications apply; in particular, we assume N is very large.

The value of C such that $P_I = P_{II}$ is that value for which either integral assumes the value ½. At this value of C, the first quartile of x = n_0 – N/2 falls at (C-N)/2. This is in rough agreement with the results shown in Figs. 10.

This zeroth order estimate of the position of the transition assumes agents playing a random choice game. One of the dynamical effects of including information use in the game is that the position of the coexistence region moves somewhat with m. In Fig. 10, we concentrated on the existence of the coexistence region and its disappearance as m decreases. To show that the precise position of the coexistence region also depends on m, Fig. 11 plots σ^2/N for C=30, as a function of N and m, for values of N somewhat larger than the minority game configuration. Notice that the position of the minimum depends on m, moving out to larger N as m increases. It is not hard to understand, qualitatively, why this should be so. Consider approaching the coexistence region from the large N side. Very roughly, (-,-) states of the system dominate. Suppose there is some small probability that the system, at some time step is in a (+,-) or (-,+) state. In order for the system to remain in the (-,-) subspace at a given time step, it is necessary for the system to have been in a (-,-) state for each of the preceding m time steps. (Remember that the strategies respond to inputs consisting of system responses of the last m time steps.) Absent strong emergent effects to the contrary, as m increases for fixed N and C, it becomes more likely that the system will *not* have been in the (-,-) state for all preceding m time steps. In order to maintain the probability of staying in the (-,-) subspace as m increases, one needs to reduce the probability that the system is in a (+,-) or (-,+) state at any given time step. For fixed C, this can be accomplished by increasing N. Thus, for N>C, we expect to see systems with similar emergent dynamics at increasing N as we increase m, and so, for N>C, we expect the coexistence region to move out in N as m increases. This is consistent with the results shown in Fig. 11.

Of course, these are only crude arguments for the position of a cross-over region and do not address any of the dynamics in the cross-over regions. Remember also, that in both the coexistence branches there is emergent coordination (albeit of different types), and this is ignored in this simple argument. Nevertheless, our simple arguments do clearly embody the basic idea of a cross over as the system moves from strategy sub-spaces dominated by (+,-) and (-,+) to those dominated by (-,-) or by (+,+).

v. Summary of the Phase Structure

Fig. 12 presents what we believe to be the most important result of this work: A schematic phase diagram of the various behaviors of the resource allocation game as a function of the relative load on the system, and the amount of information being used by the agents to make their choices. This figure is drawn for fixed C. Load, on the vertical axis, is represented by N, and the amount of information is represented on the horizontal axis by m. Note in particular the existence of the coexistence region and its disappearance at m_c^*. As we have alluded to earlier, there is a more general and robust way to plot this figure, as a function of intensive dynamical variables, so that the artificial difference between m_c and m_c^* disappears and the phase transitions for the minority game configuration, N=C+1 and other configurations occur at the same value of the information variable. This will be discussed at length elsewhere.[4]

C. Comparisons Between Binary Satisfaction and Partial Satisfaction

In our discussion of the results of the resource allocation game, we have assumed a simple binary payoff function. We have also explored the game assuming partial satisfaction, in which an overloaded supplier provides a fractional amount of resource, C/(2n), to each of the n agents using that supplier at that time step. Strategies are similarly rewarded, as explained in detail in section II. Here we briefly compare the results of the resource allocation game played with binary and partial payoff schemes.

The general structure of the games played with partial satisfaction is quite similar to the three-dimensional graphs shown in Figs. 2-5. There are four points of comparison that are particularly noteworthy to stress. First, unlike the case of binary satisfaction, there is no precise symmetry between the games with a given N and with C=N-1±d. (The symmetry for binary satisfaction is proved in the Appendix.) Nevertheless, empirically we find that the values of σ^2/N and η in games played with these two values of C are quite similar in the case of partial satisfaction. Second, unlike the case of binary satisfaction, there is no smooth region in σ^2/N for N>>C or N<<C. The reason is, of course, that since partial credit is awarded to an agent's strategies, the standard deviation in the number of agents using a given supplier will not follow a simple random process. We will no longer have N/2 agents with tied strategies that must make a decision of which group to join by a coin flip. Rather, it will be rare to have a pair of tied strategies, and the number of agents using a given supplier at a given time will depend on which strategies happen to be more highly ranked at that time, which is in turn a function of the particular history of n_0 and n_1 in that particular game. The variation in this process within a single game and among games is generally much higher than a simple random choice among N/2 agents. Thus, σ^2/N will not be ~1/8, and η will not be small. For the same reason, much of the fine structure we observe when N and C have values that are fairly far away from the diagonal, N=C+1, (i.e. past the valley in σ^2/N) is missing in the case of partial satisfaction. Specifically, σ^2/N for different games does not, primarily, take on values that are integral multiples of 1/16, as in the case of binary satisfaction. Rather, the distribution of σ^2/N smoothly broadens as N increases for fixed C. A typical result is shown in Fig. 13. The occurrence of values that are multiples of 1/16 depends on many agents with tied strategies who must choose which supplier to use by flipping a coin, and that does not generally occur with partial satisfaction.

Fourth, and perhaps most importantly, the bifurcated coexistence region marking the phase change from competition for limited resources to scarce (or abundant) resources does continue to exist in the case of partial satisfaction. This is a very important observation, since it tells us that this phase change is robust, and does not occur only in a very simple, relatively contrived game. We anticipate, in fact, that a phase change, possibly accomplished by passage through a coexistence region, is a fairly general feature of resource allocation games. Thus, in general, we expect that, in resource allocation problems, as we increase demand for fixed supply, the system passes from one well defined phase to another, with some evidence of non-analyticity.

IV. Discussion

In this paper we have studied the behavior of systems in which agents compete to obtain resources from one of two suppliers, each of which has a restricted resource supply. We have seen that this class of systems, which includes the minority game as a special case, has an extremely rich set of behaviors as control parameters such as N, C and m are varied.

Perhaps the most important findings of our work are the existence of phase changes with fairly well-defined coexistence regions as a function of N and C and the disappearance of coexistence as m approaches m_c^* from above. These phase changes are in addition to the phase change as a function of m, studied most completely in the context of the minority game. The new phase changes we have discovered separate, as a function of N and C, regions in which agents compete for limited resources from regions in which there are scarce or abundant resources. The former regions, which encompass a range of values of C for fixed N, exhibit dynamics qualitatively similar to that seen in the minority game (C=N-1), while the regions of scarce and abundant resources have quite different dynamics. It is important to emphasize that the existence of this phase change, and in particular its accomplishment through a well-defined bifurcated coexistence region, appears to be robust, in that it is seen both in games with binary satisfaction, and in games with partial satisfaction.

For large enough values of m ($>m_c^*$) the bifurcated coexistence region consists of two bands. In one, games evidence behavior qualitatively similar to that shown by games in the limited resource region, while in the other, games show behavior similar to that of games in the scarce or abundant resource region (depending on whether N>C or N<C).

Although one might have surmised that the initial distribution of strategies in a given game would have determined the branch in which that game would be located, this appears not to be the case. We have performed experiments in which we have kept the distribution of strategies the same altering 1. the initial m-history of the system, and 2. the random number seed which determines the sequence of quasi-random numbers that will be used to determine how ties between strategies are broken. We have found that both these variables, in addition to the initial distribution of strategies, are important in determining which branch a given game will occupy in the coexistence region.

A very interesting question is what happens when evolution is introduced into the games, particularly with regard to this phase change and the coexistence region. In all the games considered here, strategies that are used by the agents do not change over the course of the game, and it is unclear what the fate of this coexistence region will be if the agents are allowed to evolve their strategies to improve performance. It is possible that the bifurcated coexistence structure will survive, or that the nature of the phase change will alter once evolution is introduced. This is an extremely important question. Evolution of strategies is expected to be a feature of many real systems involving resource competition, and so it is important to understand as generally and realistically as possible, how systems behave as resources are made increasingly scarce or increasingly abundant.

In addition to this phase change, there are a number of other interesting features seen in these games, particularly in the game with binary satisfaction. These include a transition from a region in which σ^2/N is smooth as a function of N and C (for N>>C or N<<C) to a region in which σ^2/N is considerably more variable as a function of N and C. The transition between these two behaviors is also accompanied by coexistence regions and other interesting structures. However, as intriguing as these features are, we consider them less significant than the phase change that takes us from limited to scarce or abundant resources. The reason is that these other features are apparently not very robust. They appear to be strongly dependent on the binary satisfaction payoff scheme and are largely absent in games played with partial satisfaction.

In any case, we believe that it is particularly important to study games in which the balance between supply and demand is not too far from the minority game configuration. In general, we expect that many systems in the real world will evolve toward a state in which resources are, in some sense, balanced with demand. So, for example, if there is a high demand for personal computers, more companies will form to produce them and existing companies will increase their output. As demand diminishes, companies will go out of business, and those that survive will reduce output, driving the system toward some general state in which supply is appropriate for the demand. In the minority game, supply and demand are frozen into the system in a way that enforces competition for limited resources. In the real world, though, this balance is not imposed by fiat, but generally occurs dynamically. In this paper, we have not studied games in which the dynamics of emergent balance between supply and demand plays a role. (This will be examined in more detail elsewhere.) However, we have taken a step in that direction by relaxing the special constraint imposed in the minority game between supply and demand, to see how systems with other supply/demand ratios behave.

Earlier in the paper we also discussed the fact that the minority game configuration is a singular point in the space of control variables, N, C and m. As a function of these variables, games played with N=C+1 have a dramatically different outcome than games played with neighboring values of N. This is a fairly awkward circumstance facing any attempt to understand the behavior of different games with different values of N. C and m. To overcome this awkwardness (as we will discuss in more detail elsewhere[4]) it is most important that our games be considered functions of dynamically relevant variables, which may not be the same as the control variables m, N and C. We find that when we consider dependent variables such as σ^2/N as a function of more dynamically relevant variables, the minority game configuration loses its special status, and has outcomes more consist with nearby configurations. This supports our view that, seen as a function of appropriate variables, there are well defined extended phases such that games in the same phase share similar behavior, and games in different phases evidence qualitatively and distinctly different behaviors.

The behavior of a class of games that models competition for resources under various loads is rich and intriguing. Much work remains to be done to fully understand these systems. However, there are strong indications that these relatively simple games have important and robust features that can help us understand, at a deeper level, resource allocation in real systems.

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I. Figures

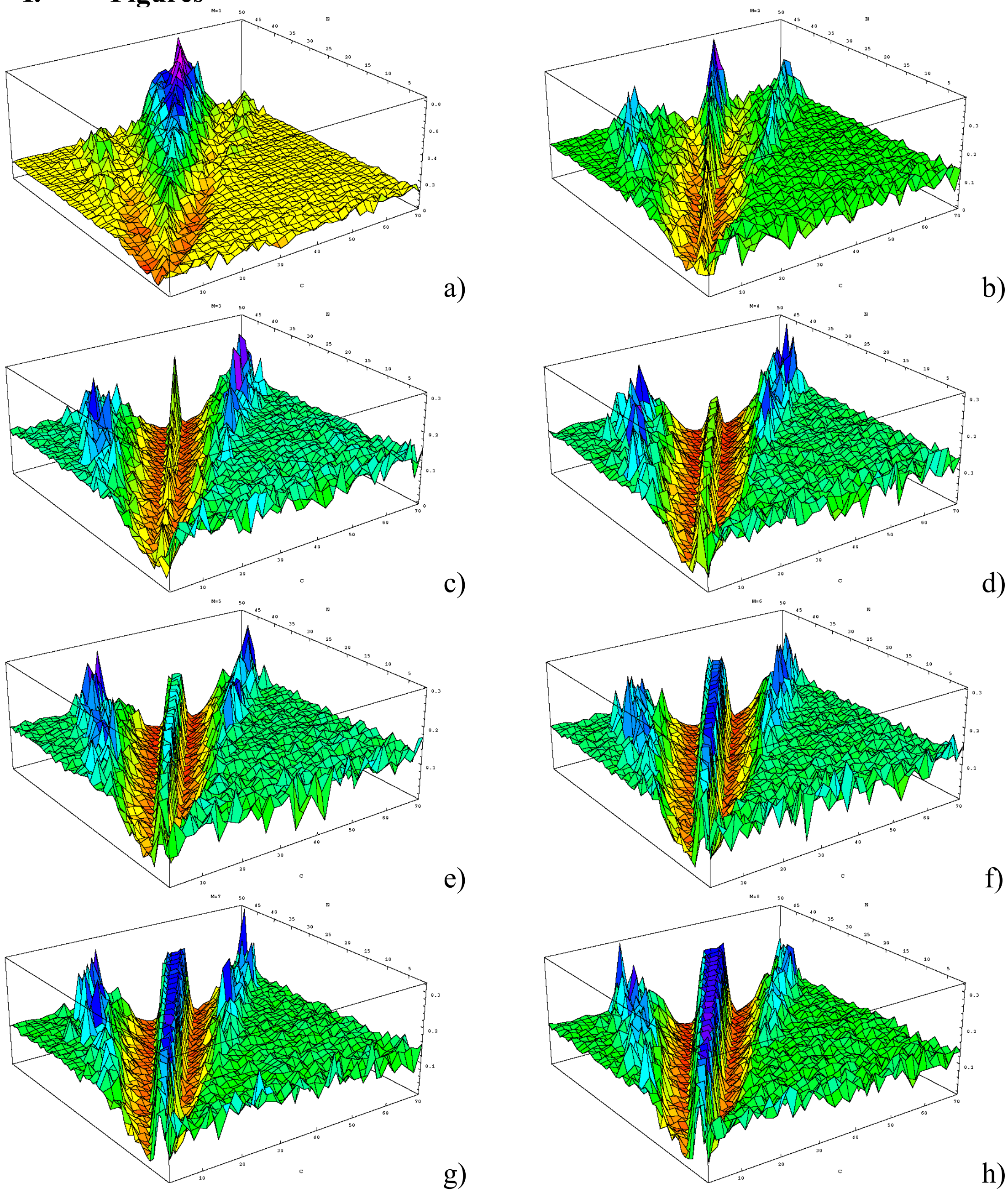


Fig. 2. A sequence of plots of σ^2/N as a function of N ($3 \le N \le 50$) and C ($2 \le C \le 70$). for fixed m. a) m=1, b.) m=2, c.) m=3, d.) m=4, e.) m=5, f.) m=6, g.) m=7, h.) m=8. Only even values of C are plotted since we have restricted our numerical studies to the case in which both suppliers have equal capacity. A larger range of C is plotted to show the d $\leftrightarrow$ -d for binary satisfaction discussed in the text.

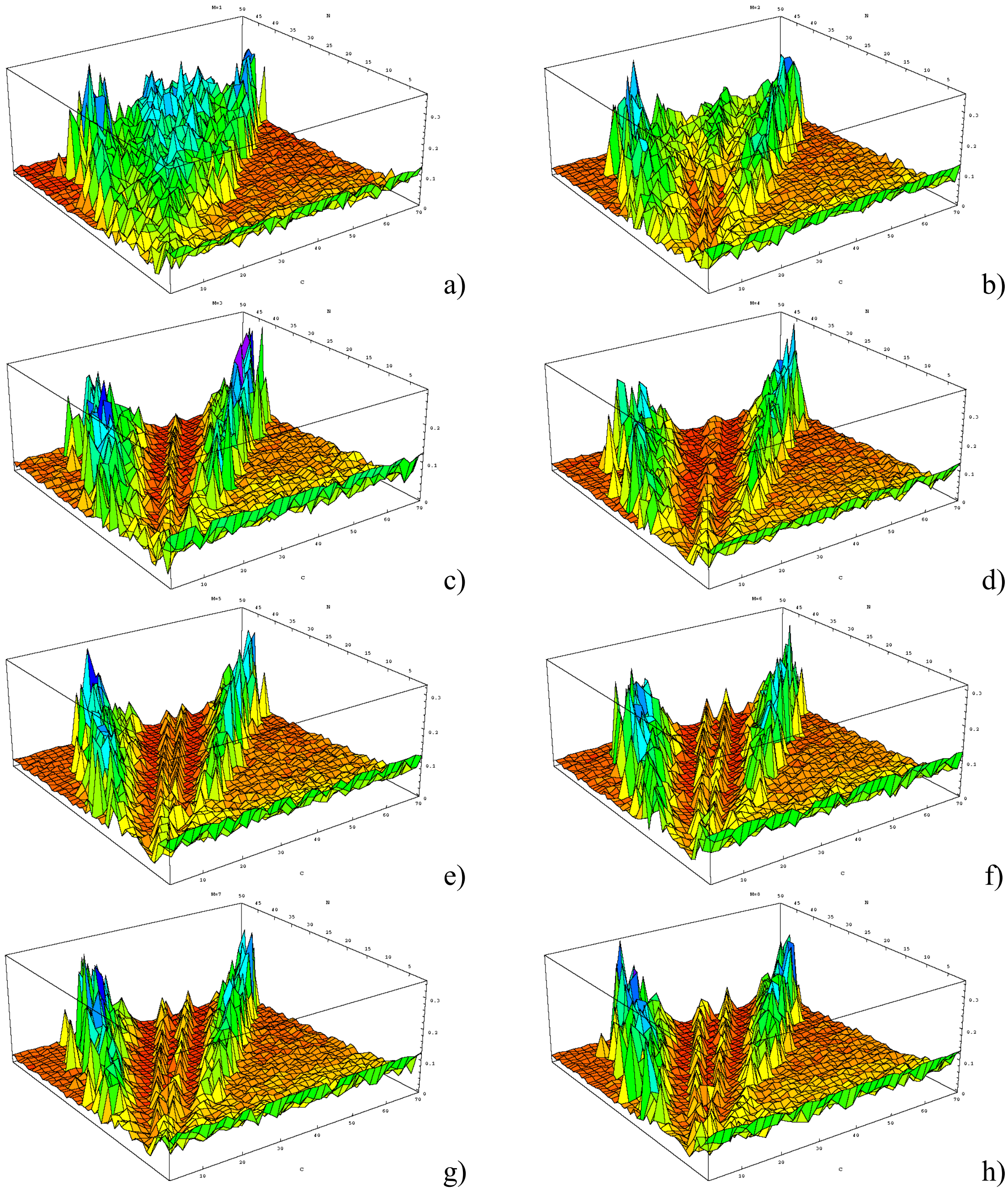


Fig. 3. A sequence of plots of $\sigma[\sigma^2/N]$ as a function of N and C for fixed m, for the same range of parameters as in Fig. 2. a) m=1, b.) m=2, c.) m=3, d.) m=4, e.) m=5, f.) m=6, g.) m=7, h.) m=8.

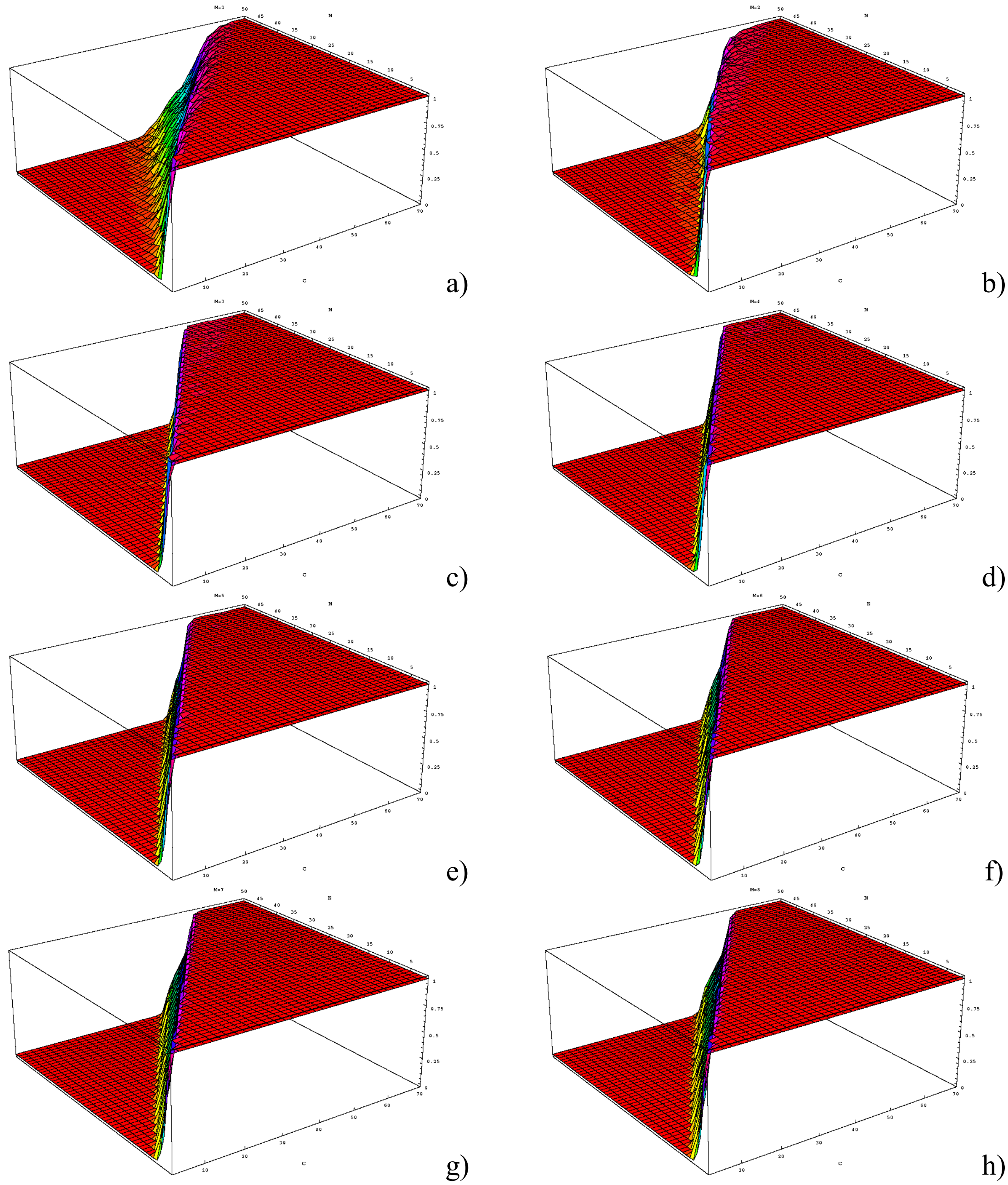


Fig. 4. A sequence of plots of mean agent wealth, W, as a function of N and C for fixed m, for the same range of parameters as in Fig. 2. a) m=1, b.) m=2, c.) m=3, d.) m=4, e.) m=5, f.) m=6, g.) m=7, h.) m=8.

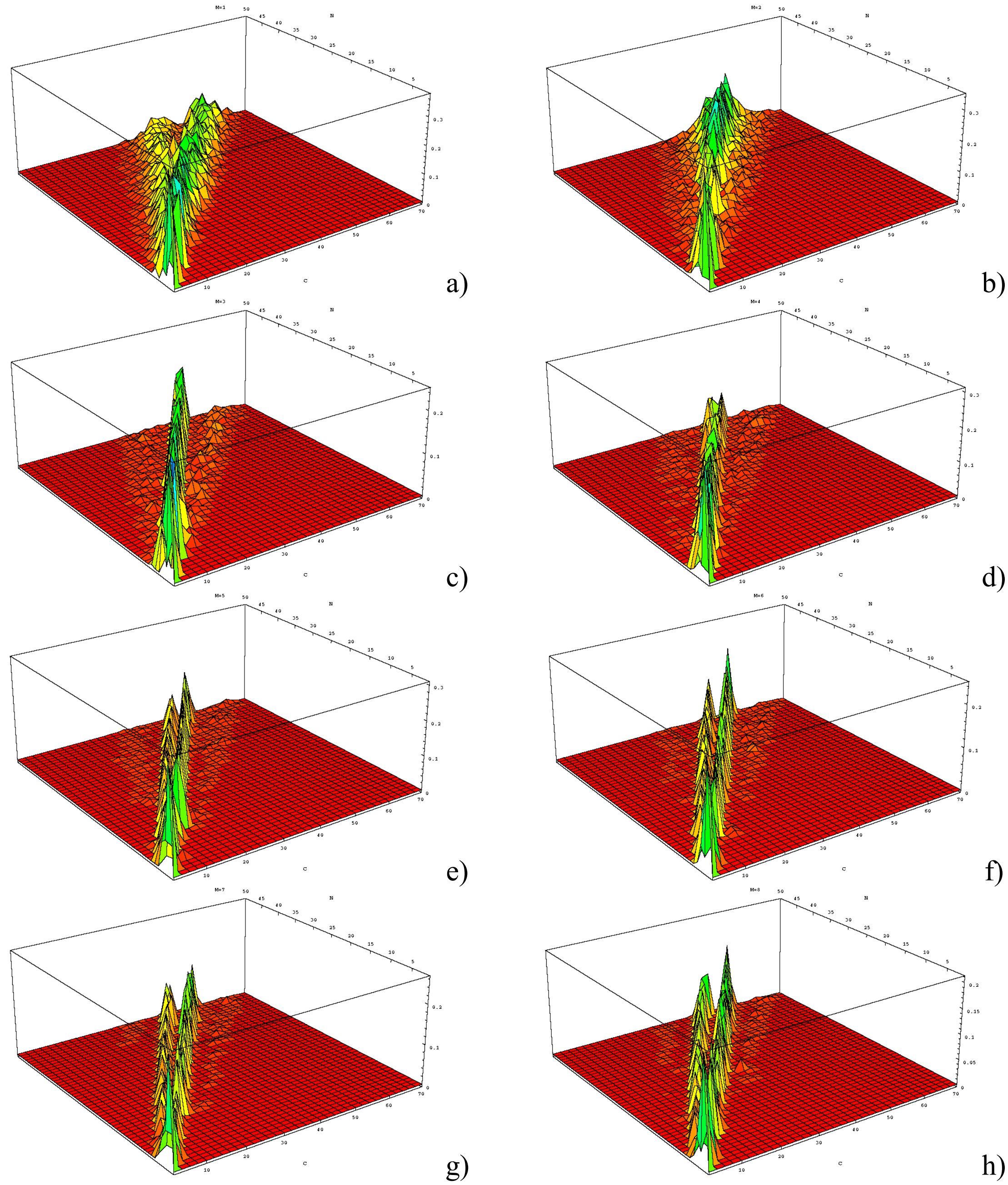
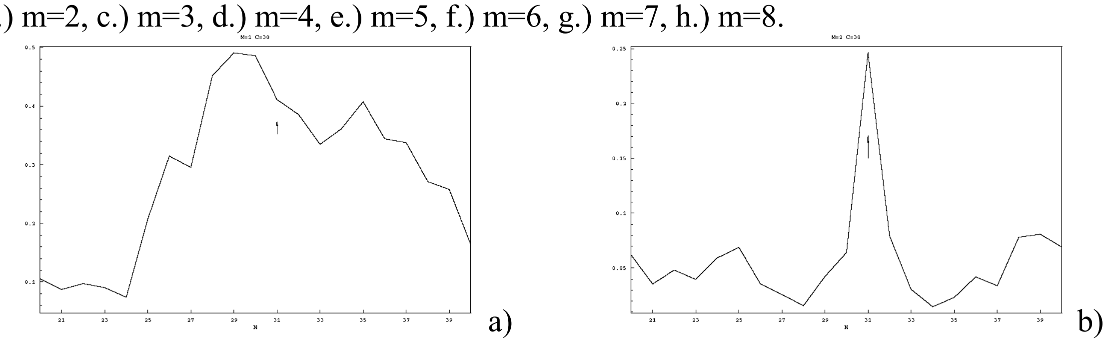


Fig. 5 A sequence of plots of the standard deviation of agent wealth, σ[W] as a function of N and C for fixed m, for the same range of parameters as in Fig. 2. a) m=1, b.) m=2, c.) m=3, d.) m=4, e.) m=5, f.) m=6, g.) m=7, h.) m=8.



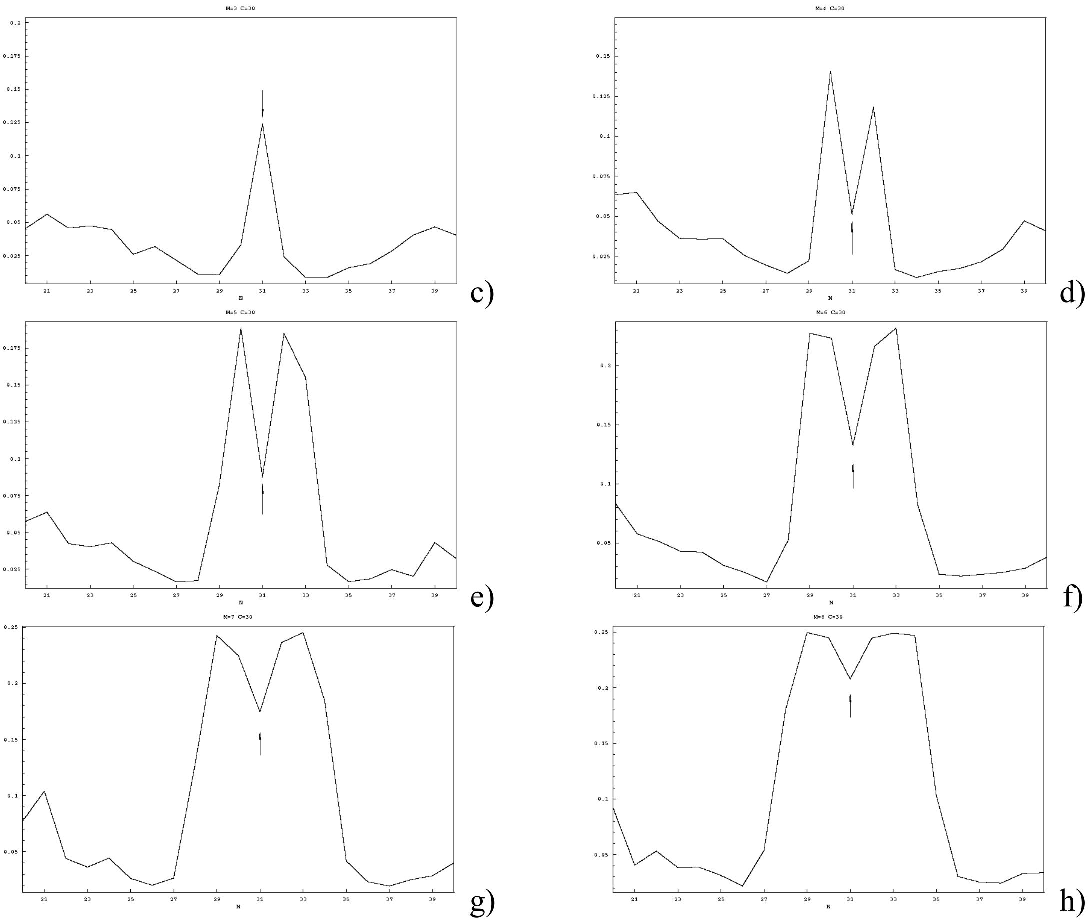


Fig. 6. σ^2/N as a function of N ($20 \le N \le 40$) for C=30, near the minority game configuration (N=C+1) for different m. The arrow in each graph indicates the result for minority game configuration. a) m=1, b.) m=2, c.) m=3, d.) m=4, e.) m=5, f.) m=6, g.) m=7, h.) m=8.

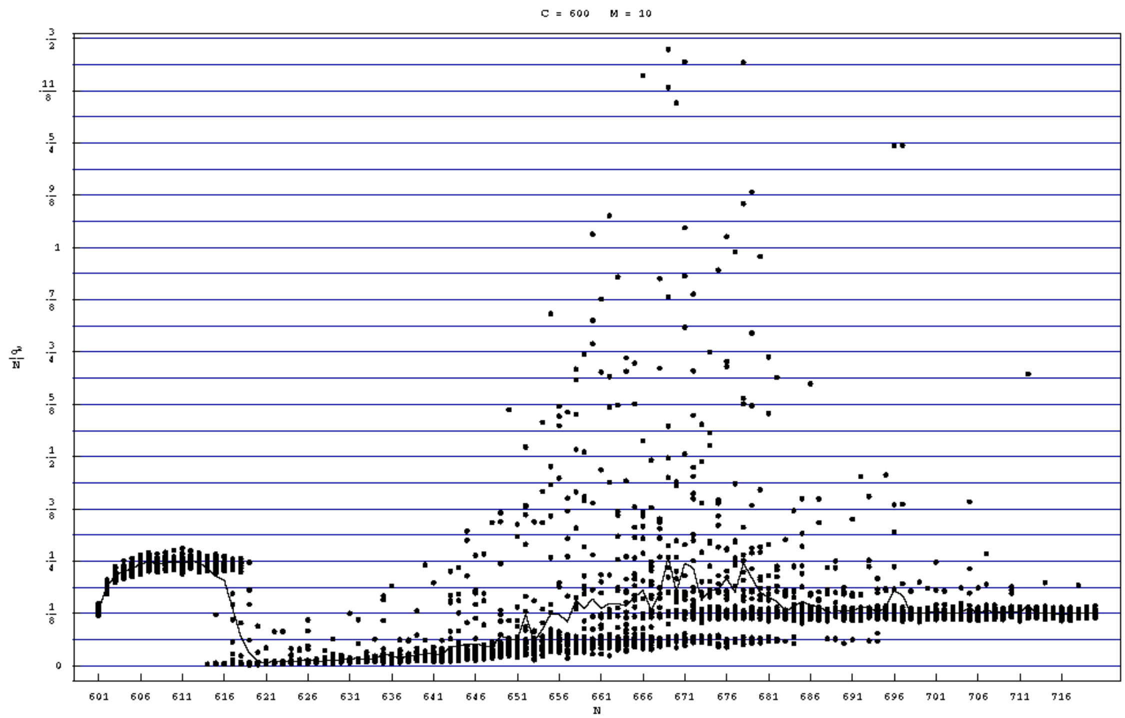


Fig. 7. σ^2/N for different runs as a function of N. ($601 \le N \le 720$) for C=600 and m=10. 32 runs are presented for each value of N. This figure illustrates the coexistence region, the bands for large N and the cloud of outliers, as described in the text.

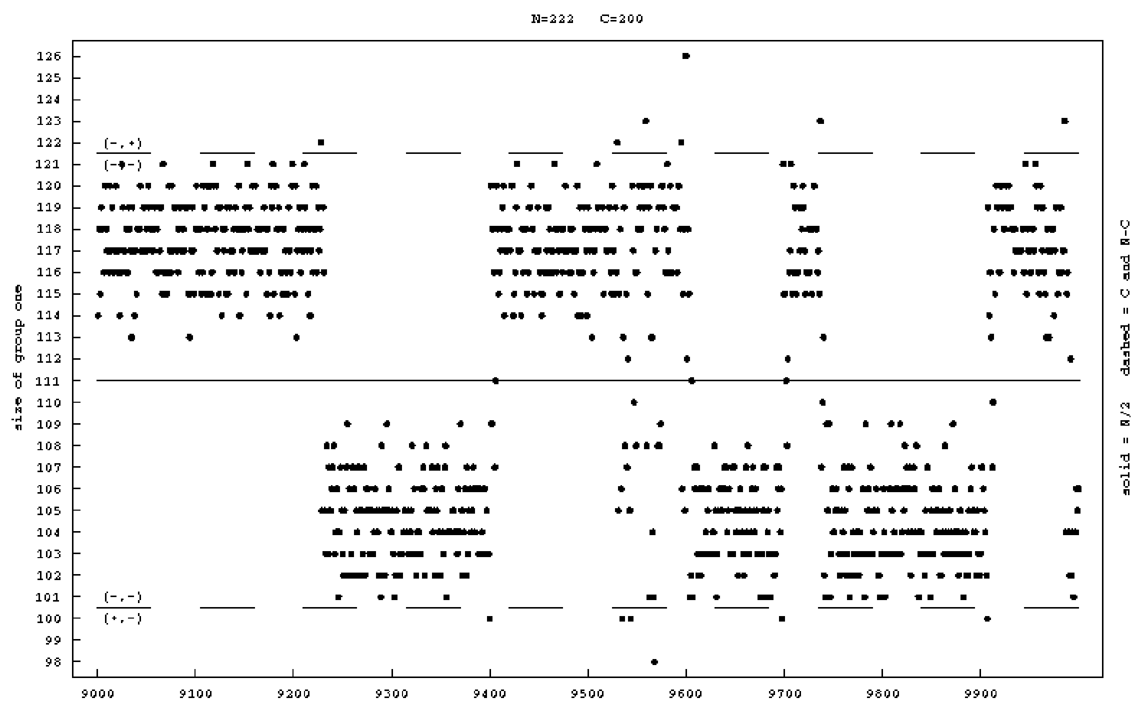


Fig. 8. A section of the time series of occupancy of group 1, n_1, for a run in the region of scarce resources with N=222, C=200 and m=6, illustrating the bursting phenomenon described in the text. The solid line indicates the value n_1=N/2, and the dashed lines

indicate n_1=C/2 and n_1=N-C/2. These lines demarcate the states (-,-) from (+,-) and (-,-) from (-,+), respectively. Note that just prior to each burst there is an event outside the (-,-) region.

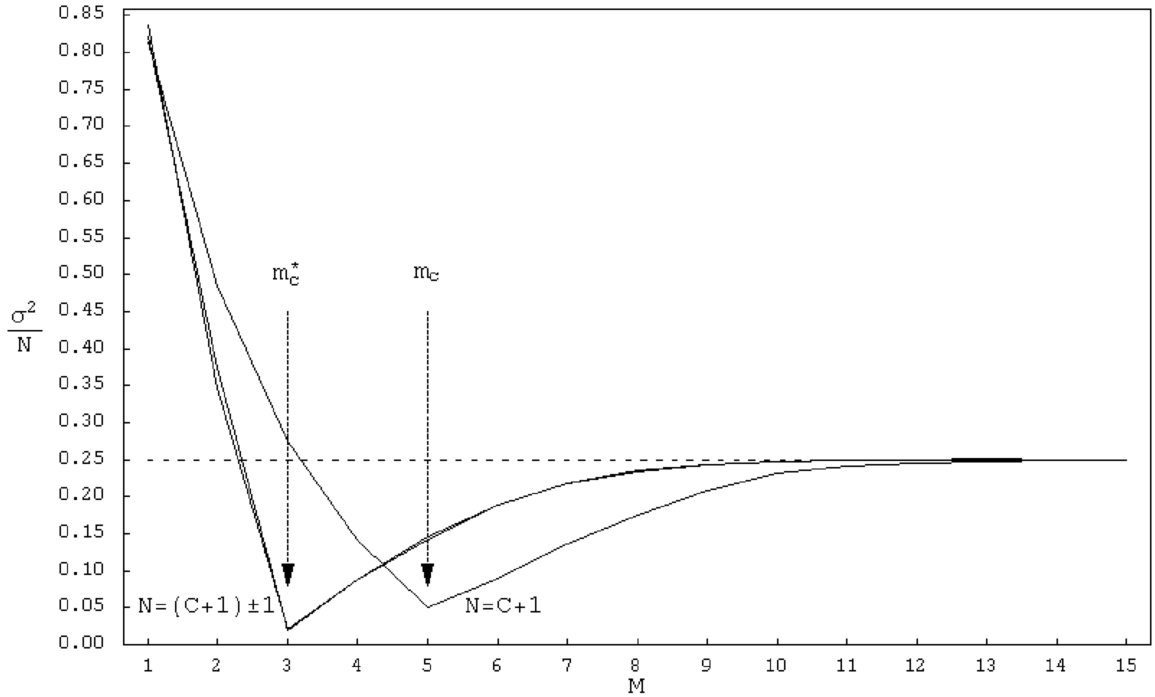
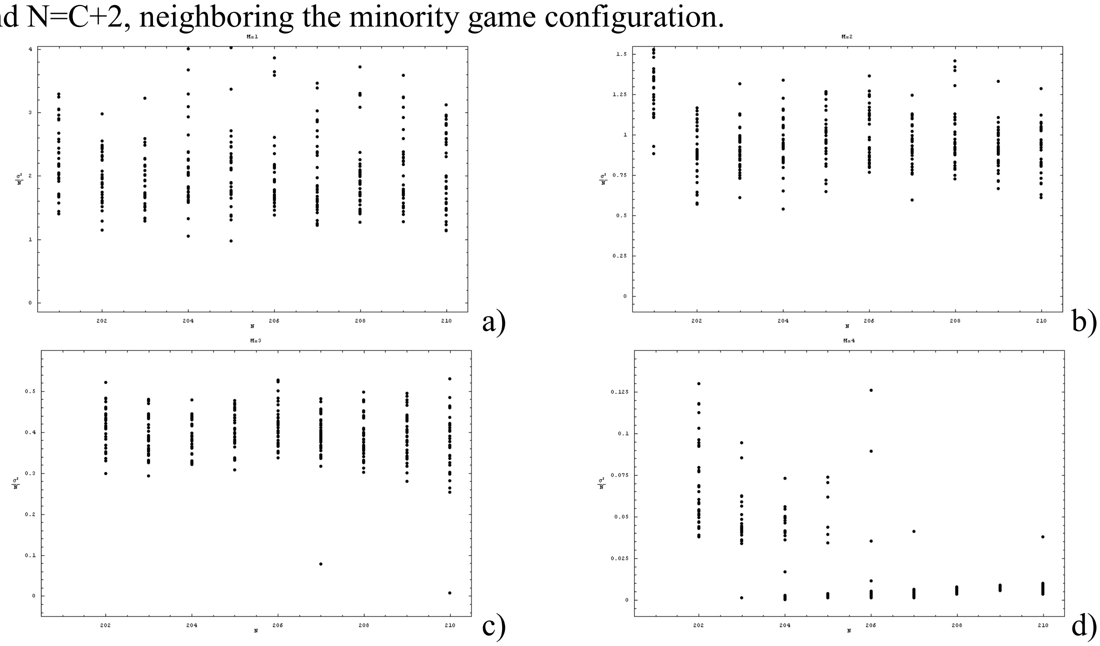


Fig. 9. σ^2/N as a function of m for N=C+1 (the minority game configuration) and N=C and N=C+2, neighboring the minority game configuration.



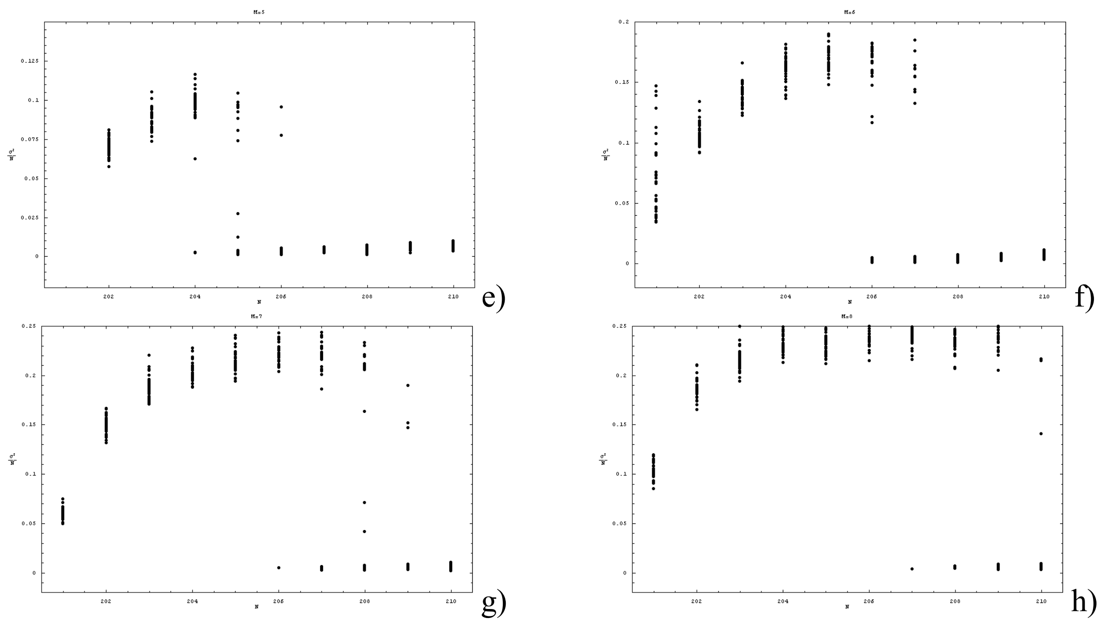


Fig. 10. A sequence of graphs of σ^2/N as a function of N ($201 \le N \le 210$) for C=200 and for various m in the transition region between limited and scarce resources. a) m=1, b.) m=2, c.) m=3, d.) m=4, e.) m=5, f.) m=6, g.) m=7, h.) m=8**.**

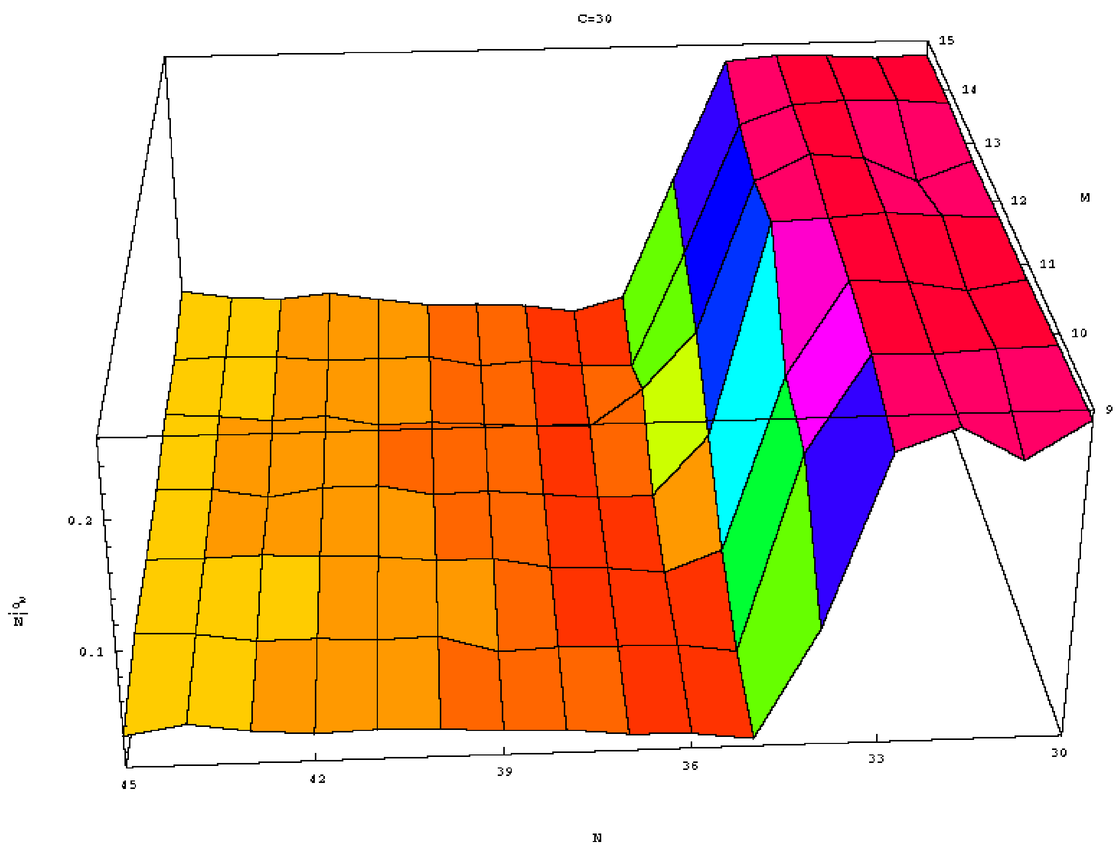


Fig. 11. σ^2/N as function of N, ($30 \le N \le 45$) and m ($9 \le m \le 15$) for C=30. Note that the position of the minimum of σ^2/N moves to higher N as m increases.

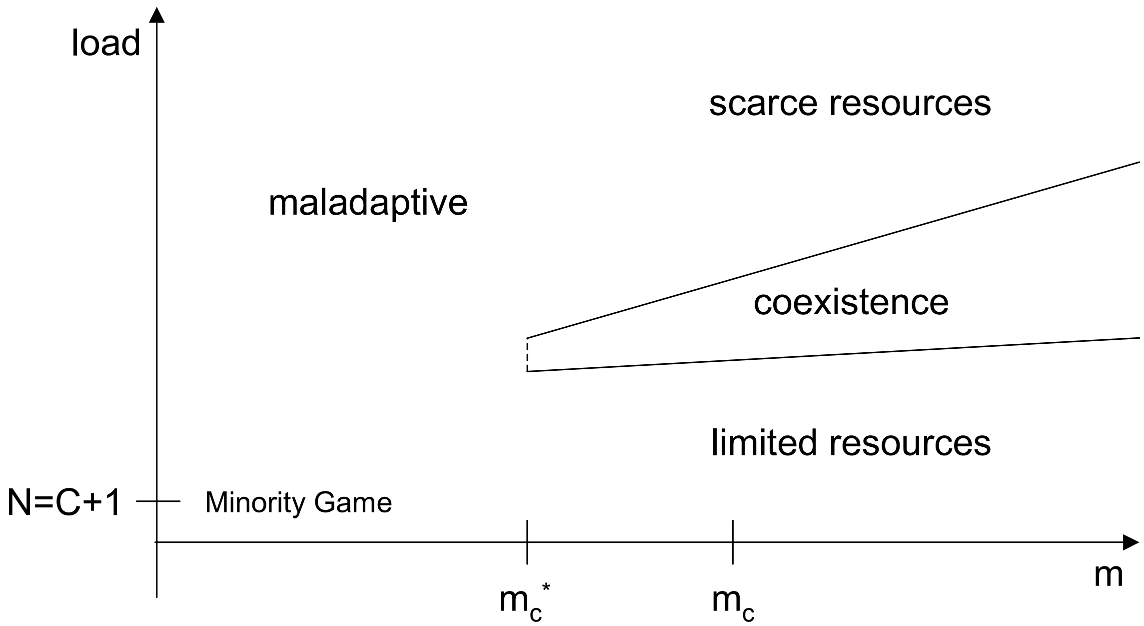


Fig. 12. A schematic phase diagram for resource allocation games as a function of relative load and the amount of information used by the agents to make their choices. As discussed in the text, this figure is drawn for fixed C so that N represents load and the amount of information is represented by m.

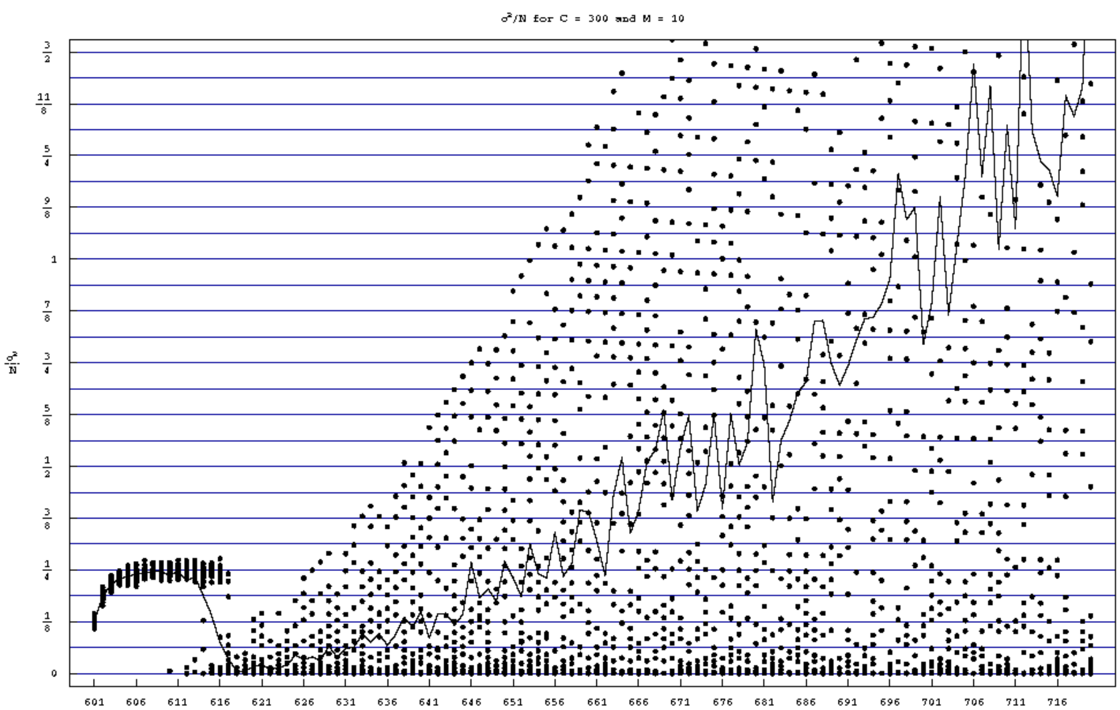


Fig. 13. σ^2/N as a function of N for fixed C=300 for the case of partial satisfaction.

II. APPENDIX

In this appendix, we prove the following theorem: Consider two resource allocation games, with binary satisfaction, each with memory m and with N agents. Call the capacities of the two suppliers in the first game, $C_0(1)$ and $C_1(1)$ and the capacities of the two suppliers in the second game $C_0(2)$ and $C_1(2)$. (Note that we consider here a somewhat more general case than that considered in the body of this paper. Namely, we consider the case here of games in which the two suppliers may have different capacities.) Let $C(1)=C_0(1)+C_1(1)$ and $C(2)=C_0(2)+C_1(2)$. Now suppose that $C(1)= C_0(1)+C_1(1)=N-1-d$ and $C(2)=N-1+d$ with $C_0(2)=C_0(1)+d$ and $C_1(2)=C_1(1)+d$. Suppose that the strategies distributed to the N agents are the same between the two games. Then, the time series of group occupancy in these two games is identical for any d. (Technical note: For this theorem to obtain, ties between strategies must be broken in the same way. So, for example, if ties are broken by a deterministic algorithm (eg. in case of a tie, the strategy used last is used) that algorithm must be the same. If ties are broken according to a pseudo-random coin flip, the same random number generator with the same random seed must be used. Also, we assume that the initial (random) m-string of minority groups used to start the game is the same.)

The proof of this assertion rests on the observation that for a given occupancy in the two groups, n_0 and n_1, changes in the strategy rankings between two strategies used by a given agent are the same in the two games described above. Consequently, at the same given time in each of the two games, a given agent will use the same strategy. Each agent's decision will be the same at corresponding time steps in the two games, and so group composition will be identical as a function of time. Note that this theorem tells us that the time series of group membership is the same. Consequently, quantities like σ^2/N and η will be identical in the two games. However, wealth and wealth distribution will, in general, be quite different.

Consider, then, a distribution of agent choices at some time of game 1 which leads to group occupancy of $n_0(1)$ and $n_1(1)$. There are several cases to consider:

I. $n_0(1) \le C_0(1)$ and $n_1(1) \le C_1(1)$

II. $n_0(1) \le C_0(1)$ and $n_1(1) > C_1(1)$

III. $n_0(1) > C_0(1)$ and $n_1(1) \le C_1(1)$

IV. $n_0(1) > C_0(1)$ and $n_1(1) > C_1(1)$

In cases I and IV, there is no change in relative strategy rankings, because each strategy, regardless of whether it predicted 0 or 1 in response to the current m-history, will receive the same award: 1 point in case I and 0 points in case IV. In case II. a strategy that responds 0 will gain a point over a strategy that responds 1, while in case III a strategy that responds 1 will gain a point over a strategy that responds 0. Depending on the value of d, one or more of these cases may not be realizable in game 1. Without loss of generality, we take $d>0$.

We now show that for each of these cases that are accessible for a given value of d, the same group occupancy n_0 and n_1 results in the same strategy awards in game 2.

Case I. $n_0(1) \le C_0(1)$ and $n_1(1) \le C_1(1)$. If $d>0$, this case is not accessible since $C_0(1)+C_1(1)=N-1-d<N$, but according to Case I, $C_0(1)+C_1(1)\ge N$, a clear contradiction.

Case II. $n_0(1) \le C_0(1)$ and $n_1(1) > C_1(1)$. Now, since $C_0(2)=C_0(1)+d$ and $C_1(2)=C_1(1)+d$, and since $d>0$, $n_0(1) \le C_0(1) \le C_0(2)$ Also, $n_1(1)=N- n_0(1)\ge N-C_0(1)=C_0(1)+C_1(1)+1+d-C_0(1)= C_1(1)+1+d=C_1(2)+1$. Thus, $n_0(1) \le C_0(1)$ and $n_1(1) > C_1(1) \Rightarrow n_0(1) \le C_0(2)$ and $n_1(1) > C_1(2)$. Therefore, in this case the group occupancies result in the same awards to strategies with the same predictions in the two games.

Case III. $n_0(1) > C_0(1)$ and $n_1(1) \le C_1(1)$. Since $C_0(2)=C_0(1)+d$ and $C_1(2)=C_1(1)+d$, and since $d>0$, $n_1(1) \le C_1(1) \le C_1(2)$ Also, $n_0(1)=N- n_1(1)\ge N-C_1(1)=C_0(1)+C_1(1)+1+d-C_1(1)= C_0(1)+1+d=C_0(2)+1$. Thus, $n_1(1) \le C_1(1)$ and $n_0(1) > C_0(1) \Rightarrow n_1(1) \le C_1(2)$ and $n_0(1) > C_0(2)$. Therefore, in this case the group occupancies result in the same awards to strategies with the same predictions in the two games.

Case IV. $n_0(1) > C_0(1)$ and $n_1(1) > C_1(1)$. Since $C_0(2)=C_0(1)+d$ and $C_1(2)=C_1(1)+d$, and since $d>0$, $n_0(1) = N-n_1(1) = C_0(1)+C_1(1)+1+d-n_1(1) < C_0(1)+C_1(1)+1+d-C_1(1) = C_0(1)+1+d = C_0(2)+1$. Similarly, $n_1(1) = N-n_0(1) = C_0(1)+C_1(1)+1+d-n_0(1) < C_0(1)+C_1(1)+1+d-C_0(1) = C_1(1)+1+d = C_1(2)+1$. Thus, $n_0(1) > C_0(1)$ and $n_1(1) > C_1(1) \Rightarrow n_0(1) < C_0(2)$ and $n_1(1) < C_1(2)$. So, with these occupancy numbers, no strategy gets a point in game 1, regardless of which group it predicts, while in game 2 all strategies get a point. Therefore, in this case the relative rankings of the two strategies are always the same in the two games.

Note that this proof tells us, not only that the time series of group occupancy is the same in both games, but that the time series of the choices made by each agent is the same. Unfortunately, the same detailed proof does not obtain in the case of partial satisfaction. In that case, awards to the agents strategies is not just a binary function of the n_i, but vary continuously with the n_i relative to the C_i, and so awards to the strategies will, in general differ in the two games. Only for special, relatively contrived partial satisfaction payoff functions will this proof obtain. That said, it is nevertheless the case that σ^2/N and η are generally quite similar in the games that lie symmetrically from the minority game configuration ($d \rightarrow -d$).