eppstein k-best

knight
Push A
Pop A & go through A’s arcs
  Push B.1 (from A)
  Push E.50 (from A)
Pop B & go through B’s arcs
  Push C.3 (from B)
  Push D.18 (from B)
Pop C & go through C’s arcs
  Push Q.7 (from C)
  Adjust D.6 (from C)

Analysis:
  total pops = n
  total pushes = m
  maxheapsize = n
  cost of pop = log n
  cost of push = 1
  O(m + n log n)
K-best

- Eppstein complexity is $O(m + n \log n + k \log k)$
- Great to keep $n$ and $k$ separate – both might be 100,000!
- But consider:

  length of 1-best path: $n$
  length of 2-best path: $2n$
  ...
  length of $k$-best path: $kn$

  total of all these lengths = $n(1+\ldots+k) = O(n \, k^2)$

- How can Eppstein do better? Well, printing paths is your problem.
- Eppstein computes “implicit representation of $k$ best paths in $O(m + n \log n + k \log k)$ time … from which we can read off the $k$th best path in time proportional to the number of edges in that path… and we can read off the length of that path in constant time.”
- $k$-best path lengths problem solvable in $O(m + n \log n + k \log k)$
K-best

- Normal case for speech, MT, and NLG: k-best entries have roughly the same lengths
- Or even:

  \[
  \begin{align*}
  \text{length of 1-best path:} & \quad 0 \\
  \text{length of 2-best path:} & \quad 1 \\
  \text{length of 3-best path:} & \quad 1 \\
  \text{length of 4-best path:} & \quad 2 \\
  \text{length of 5-best path:} & \quad 2 \\
  \text{length of 6-best path:} & \quad 2 \\
  \text{length of 7-best path:} & \quad 2 \\
  \text{length of 8-best path:} & \quad 3 \\
  \text{length of 9-best path:} & \quad 3 \\
  \end{align*}
  \]

- Summed lengths of k-best paths = \( O(k \log k) \), not \( k^2 \) [homework]
- As \( k \) goes from 1m to 1b, lengths only increase from 20 to 30
Eppstein
Eppstein

Best path = 55
First Try: Breadth-First Search

Maintain a heap of partial paths. Keep popping and pushing. When a complete path is popped, print it. Vast amount of work even before the 1-best.
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Core Eppstein idea:

Make a heap where all elements are complete paths. Every pop gives you the next longest complete path.
Eppstein

Best path = 55
Every node has its own best edge
Best path = 55
Every node has its own best edge
Every node has its best cost to goal
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Every node has its own best edge
Every node has its best cost to goal

What if we took this edge instead of 13? We’d wind up with a worse path, worse by 2 + (56-55) = 3.

What if we took this edge instead of 12? Worse by 25 + (7-23) = 9.
Best path = 55
Every node has its own best edge
Every node has its best cost to goal

Every path can be represented as a sequence of sidetrack edges.

So \{\}, \{3\}, \{9\}, \{3,1\}, \{3, 4, 9\}, etc. all correspond to paths in original graph.
Eppstein

Best path = 55
Every node has its own best edge
Every node has its best cost to goal

Every path can be represented as a sequence of sidetrack edges.

So \{\}, \{3\}, \{9\}, \{3,1\}, \{3, 4, 9\}, etc. all correspond to paths in original graph.
Taking yet each successive sidetrack incurs some incremental cost.

Cost of sidetrack sequence is sum of those incremental costs.

Now, enumerate sidetrack sequences in order of cost. (& add 55 to each sequence’s cost!)
Algorithm (Breadth-First Search):
push root.0 onto heap H
for i = 1 to k
    pop node r.c1 from top of H
    print cost c1 + 55
    for j = each edge from r → s with cost c2
        push node s.(c1+c2) onto H
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Algorithm (Breadth-First Search):
push root.0 onto heap H
for $i = 1$ to $k$
    pop node $r.c_1$ from top of $H$
    print cost $c_1 + 55$
    for $j = \text{each edge from } r \rightarrow s \text{ with cost } c_2$
        push node $s.(c_1+c_2)$ onto $H$
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Analysis:
total pops = k
total pushes = km (if fully connected)
maxheapsize = O(km)
cost of pop = O(log km) = O(log k + log m)
cost of push = 1
  → O(km + k log k + k log m)
  → O(km + k log k)
  → km = kn^2 = bad term to have!!
Algorithm (Breadth-First Search):
push root.0 onto heap H
for $i = 1$ to $k$
  pop node r.c1 from top of H
  print cost c1 + 55
  for $j =$ each edge from r → s with cost c2
    push node s.(c1+c2) onto H

Analysis for bounded outdegree tree:
total pops = $k$
total pushes = $k \times$ constant
maxheapsize = $O(k)$
cost of pop = $O(\log k)$
cost of push = 1
→ $O(k + k \log k)$
→ $O(k \log k)$, no $n$ factor
Algorithm (Breadth-First Search):
push root.0 onto heap H
for i = 1 to k
    pop node r.c1 from top of H
    print cost c1 + 55
    for j = each edge from r → s with cost c2
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So we might be able to reduce the outdegree…

Goals:
- bounded outdegree sidetrack tree
- maintain 1-to-1 correspondence of
  - sidetrack tree paths
  - original graph start→final paths
1. Pick best sidetrack of each node from start to final. Arrange all in binary heap.
2. Repeat for other nodes besides start.
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2. Repeat for other nodes besides start.
3. Record incremental costs.
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4. Connect each sidetrack node to the heap of its tail. Outdegree = 3.
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5. Erase node labels & add special start “+0” node.
This new tree has the same 10 paths!
But it has max outdegree = 3.
Now same algorithm (BFS) works better!

DONE!
What if a node has more than one sidetrack coming out?

These are the 2\textsuperscript{nd}, 3\textsuperscript{rd}, 4\textsuperscript{th}, etc best sidetracks leaving the start node itself. Need to add this fourth outdegree for all nodes in the new structure.
What if a node has more than one sidetrack coming out?

These are the 2nd, 3rd, 4th, etc best sidetracks leaving the start node itself. Need to add this fourth bit of outdegree for all nodes in the new structure.

What if a node has more than one sidetrack coming out?

Even these nodes need cross edges.

These are the 2nd, 3rd, 4th, etc best sidetracks leaving the start node itself. Need to add this fourth bit of outdegree for all nodes in the new structure.
Notice that 6 and 9 appear many times. Heaps need to be built so that they share a bunch of sub-structure. Done carefully, the whole structure can be built in $O(m + n \log n)$ time, leaving $O(k \log k)$ for extracting the items using BFS.
Detail #3
loops are no problem

Original graph

Sidetrack edges

Tree of sidetrack sequences

finite-representation-of
K-Best Dijkstra v. Eppstein

K-Best Dijkstra

Push (S,0)
Pop (S,0)
Do m+1 pushes
Pop (S,1)
Do m+1 pushes
Pop (S,1)
Do m+1 pushes
...
At least $O(km)$ work before any complete path is generated…

Eppstein

Each BFS-pop corresponds to a complete path:
1-1000, 2-1000, 1-1-1000, 3-1000, 1-2-1000, 2-1-1000, 1-1-1-1000, 4-1000, 3-1-1000, 1-3-1000, 2-2-1000, 1-1-2-1000, …
the end