

Optimal Choice of Granularity In Commonsense Estimation: Why Half-Orders of Magnitude

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Abstract

It has been observed that when people make crude estimates, they feel comfortable choosing between alternatives which differ by half-order of magnitude (e.g., were there 100, 300, or 1,000 people in the crowd), and less comfortable making a choice on a more detailed scale, with finer granules, or on a more spacious scale (like 100 or 1,000). In this paper, we describe two models of choosing granularity in commonsense estimates, and we show that for both models, in the optimal granularity, the next estimate is 3-4 times larger than the previous one. Thus, these two optimization results explain the commonsense granularity.

1 Introduction

People often need to make crude estimates of a quantity: e.g., estimate the size of the crowd, or someone's salary, etc. When people make such crude estimates, they usually feel reasonably comfortable choosing between alternatives which differ by half-order of magnitude. For example, a person can reasonably estimate that the size of the crowd was closer to 100, or to 300, or to 1,000. If we ask for an estimate on a more refined scale (with finer granules), or on a more

spacious scale, with larger granules, then it is much more difficult to make such an estimate (for details, see, e.g., [1]).

This observation naturally leads to the following question: What is the *optimal* granularity in commonsense estimates?

In this paper, we describe two different models for commonsense estimation, and we show that in both models, the optimal granularity is in good accordance with the observed half-order of magnitude granularity. Thus, we provide a theoretical explanation for this half-order phenomenon.

2 Gaussian Model of Commonsense Estimation

2.1 Main Idea Behind Gaussian Model

We are interested in the situation when we estimate a quantity which can only take non-negative values. To estimate the values of this quantity, we select a sequence of positive numbers $\dots < e_0 < e_1 < e_2 < \dots$ (e.g., 1, 3, 10, etc.), and every actual value x of the estimated quantity is then estimated by one of these numbers. Each estimate is approximate: when the estimate is equal to e_i , the actual value x of the estimated quantity may differ from e_i ; in other words, there may be an estimation error $\Delta x = e_i - x \neq 0$.

What is the probability distribution of this estimation error? This error is caused by many different reasons. It is known that under certain reasonable conditions, an error caused by many different reasons is distributed according to Gaussian (normal) distribution (see, e.g., [2]; this fact – called *central limit theorem* – is one of the reasons for the widespread use of Gaussian distribution in science and engineering applications). It is therefore reasonable to assume that Δx is normally distributed.

It is known that a normal distribution is uniquely determined by its two parameters: its average a and its standard deviation σ . Let us denote the average of the error Δx by Δe_i , and its standard deviation by σ_i . Thus, when the estimate is e_i , the actual value $x = e_i - \Delta x$ is distributed according to Gaussian distribution, with an average $e_i - \Delta e_i$ (which we will denote by \tilde{e}_i), and the standard deviation σ_i .

For a Gaussian distribution with given a and σ , the probability density is everywhere positive, so theoretically, we can have values which are as far away from the average a as possible. In practice, however, the probabilities of large deviations from a are so small that the possibility of such deviations can be safely neglected. For example, it is known that the probability of having the value outside the “three sigma” interval $[a - 3\sigma, a + 3\sigma]$ is $\approx 0.1\%$ and therefore, in most engineering applications, it is assumed that values outside this interval are impossible.

There are some applications where we cannot make this assumption. For example, in designing computer chips, when we have millions of elements on

the chip, allowing 0.1% of these elements to malfunction would mean that at any given time, thousands of elements malfunction and thus, the chip would malfunction as well. For such critical applications, we want the probability of deviation to be much smaller than 0.1%, e.g., $\leq 10^{-8}$. Such small probabilities (which practically exclude any possibility of an error) can be guaranteed if we use a “six sigma” interval $[a - 6\sigma, a + 6\sigma]$. For this interval, the probability for a normally distributed variable to be outside it is indeed $\approx 10^{-8}$.

Within this Gaussian model, what is the optimal granularity?

2.2 Optimal Granularity: Informal Explanation

In accordance with the above idea, for each e_i , if the actual value x is within the “three sigma” range $I_i = [\tilde{e}_i - 3\sigma_i, \tilde{e}_i + 3\sigma_i]$, then it is reasonable to take e_i as the corresponding estimate.

We want a granulation which would cover all possible values, so each positive real number must be covered by one of these intervals. In other words, we want the union of all these intervals to coincide with the set of all positive real numbers.

We also want to make sure that all values that we are covering are indeed non-negative, i.e., that for every i , even the extended “six sigma” interval $[\tilde{e}_i - 3\sigma_i, \tilde{e}_i + 3\sigma_i]$ only contains non-negative values.

Finally, since one of the main purposes of granularity is to decrease the number of “labels” that we use to describe different quantities, we want to consider optimal (minimal) sets of intervals. Formally, we can interpret “minimal” in the sense that whichever finite subset we pick, we cannot enlarge their overall coverage by modifying one or several of these intervals. Let us formalize these ideas.

2.3 Optimal Granularity: Formal Description

In the following definitions, we will use the fact that an arbitrary interval $[a^-, a^+]$ can be represented in the Gaussian-type form $[a - 3\sigma, a + 3\sigma]$: it is sufficient to take $a = (a^- + a^+)/2$ and $\sigma = (a^+ - a^-)/6$.

Definition 1.

- We say that an interval $I = [a - 3\sigma, a + 3\sigma]$ is *reliably non-negative* if every real number from the interval $[a - 6\sigma, a + 6\sigma]$ is non-negative.
- A set $\{I_i\}$, $i = 1, 2, \dots$, of reliably non-negative intervals I_i is called a *granulation* if every positive real number belongs to one of the intervals I_i .
- We say that a granulation can be *improved* if, for some finite set $\{i_1, \dots, i_k\}$, we can replace intervals I_{i_j} with some other intervals I'_{i_j} for which

$$\bigcup_{j=1}^k I_{i_j} \subset \bigcup_{j=1}^k I'_{i_j} \quad \bigcup_{j=1}^k I_{i_j} \neq \bigcup_{j=1}^k I'_{i_j},$$

and still get a granulation.

- A granulation is called *optimal* if it cannot be improved.

Theorem 1. In an optimal granulation, $I_i = [a_i, a_{i+1}]$, where $a_{i+1} = 3a_i$.

(For reader's convenience, all the proofs are placed in the last section.)

So, half-orders of magnitude are indeed optimal.

3 Uniform Model of Commonsense Estimation

3.1 Motivations

In the Gaussian model, we started with a 3σ bound, and we ended up with a sequence of granules $[a_i, a_{i+1}]$ in which the boundary points a_i form an arithmetic progression: $a_{i+1} = q \cdot a_i$ and $a_i = a_0 \cdot q^i$, with $q = 3$. We could start with a bound of 2.5σ , then we would have got a geometric progression with a different q . Which value of q is indeed optimal?

To find out, let us take into consideration the fact that a granulation is not just for *storing* values, it is also for *processing* these values. Of course, when we replace the actual value by the granule to which it belongs, we lose some information. The idea is to choose q for which this loss is the smallest.

To estimate the loss, we will consider the simplest data processing possible operation: addition. If we know the exact values of two quantities A and B , then we can compute the exact value of their sum $A + B$. In the granulated case, we do not know the exact values of A and B , we only know the *granules* to which A and B belong, and we want to find out to which of the granules the sum belongs. For example, in the above half-order granulation, we know that the first room has about 10 books, the second about 30, and we want to express the total number of books in the two rooms in similar terms.

The trouble with this problem is that the sum may belong to two different granules. Let us take an example in which we use granules $[1, 3]$, $[3, 9]$, $[9, 27]$, etc. let us assume that all we know about the first quantity A is that $A \in [1, 3]$, and all we know about the second quantity B is that $B \in [3, 9]$. In this case, the smallest possible values of $A + B$ is $1 + 3 = 4$, and the largest possible value of $A + B$ is $3 + 9 = 12$. In general, the sum $A + B$ can thus take any value from the interval $[4, 12]$. So, it could happen that the sum is in the granule $[3, 9]$, but it could also happen that the sum is in the granule $[9, 27]$.

If we want the granulation to be useful, we must assign a certain granule to the sum $A + B$. Since in reality, the value $A + B$ may belong to two different granules, no matter which of the two granules we assign, there is always a probability that this assignment is erroneous. We would like to select q for which this error probability is the smallest possible.

In order to formulate this question in precise terms, we must describe the corresponding probabilities. A natural way to describe them is as follows: If all we know about A is that A belongs to a granule $\mathbf{a}_i = [a_i, a_{i+1}]$, then it is reasonable to consider all the values from this granule to be equally probable, i.e., to assume that we have a *uniform* distribution on the interval $\mathbf{a}_i = [a_i, a_{i+1}]$. Similarly, If all we know about B is that B belongs to a granule $\mathbf{a}_j = [a_j, a_{j+1}]$, then it is reasonable to consider all the values from this granule to be equally probable, i.e., to assume that we have a *uniform* distribution on the interval $\mathbf{a}_j = [a_j, a_{j+1}]$. Since we have no information about the possible dependence between A and B , it is natural to assume that A and B are independent random variables. We are now ready for the formal definitions.

3.2 Formalization and the main result

Let $a_0 > 0$ and $q \geq 2$ be real numbers, and let $a_k \stackrel{\text{def}}{=} a_0 \cdot q^k$ and $\mathbf{a}_i \stackrel{\text{def}}{=} [a_i, a_{i+1}]$.

Definition 2. For every three integers i, j , and k , we can define $P(\mathbf{a}_i + \mathbf{a}_j \in \mathbf{a}_k)$ as the probability that $A_i + A_j \in \mathbf{a}_k$, where A_i is uniformly distributed on the interval \mathbf{a}_i , A_j is uniformly distributed on the interval \mathbf{a}_j , and A_i and A_j are independent.

If, as a result of adding \mathbf{a}_i and \mathbf{a}_j , we select the granule \mathbf{a}_k , then the probability that this assignment is erroneous (i.e., that the actual value of $A_i + A_j$ is not in \mathbf{a}_k) is equal to $1 - P(\mathbf{a}_i + \mathbf{a}_j \in \mathbf{a}_k)$. For every i and j , we want to minimize this error, so we select the value k for which this error probability is the smallest:

Definition 3. For every two integers i and j , we define the sum $\mathbf{a}_i + \mathbf{a}_j$ of granules \mathbf{a}_i and \mathbf{a}_j as a granule \mathbf{a}_k for which the error probability $1 - P(\mathbf{a}_i + \mathbf{a}_j \in \mathbf{a}_k)$ is the smallest possible. The error probability E_{ij} related to this addition is then defined as this smallest probability, i.e., as

$$E_{ij} \stackrel{\text{def}}{=} \min_k (1 - P(\mathbf{a}_i + \mathbf{a}_j \in \mathbf{a}_k)).$$

Theorem 2.

- When $q \geq \sqrt{2} + 1 (\approx 2.41)$, then:
 - $\mathbf{a}_i + \mathbf{a}_i = \mathbf{a}_{i+1}$, and
 - $\mathbf{a}_i + \mathbf{a}_j = \mathbf{a}_{\max(i,j)}$ for $i \neq j$.
- When $2 \leq q < \sqrt{2} + 1$, then:
 - $\mathbf{a}_i + \mathbf{a}_i = \mathbf{a}_{i+1}$;
 - $\mathbf{a}_i + \mathbf{a}_{i+1} = \mathbf{a}_{i+1} + \mathbf{a}_i = \mathbf{a}_{i+2}$; and
 - $\mathbf{a}_i + \mathbf{a}_j = \mathbf{a}_{\max(i,j)}$ for $|i - j| \geq 2$.

It is worth mentioning that for every q , thus defined addition of granules is commutative but not associative. Indeed:

- For $q \geq \sqrt{2} + 1$, we have:
 - $(\mathbf{a}_0 + \mathbf{a}_0) + \mathbf{a}_1 = \mathbf{a}_1 + \mathbf{a}_1 = \mathbf{a}_2$, while
 - $\mathbf{a}_0 + (\mathbf{a}_0 + \mathbf{a}_1) = \mathbf{a}_0 + \mathbf{a}_1 = \mathbf{a}_1 \neq \mathbf{a}_2$.
- For $q < \sqrt{2} + 1$, we have:
 - $(\mathbf{a}_0 + \mathbf{a}_0) + \mathbf{a}_2 = \mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_3$, while
 - $\mathbf{a}_0 + (\mathbf{a}_0 + \mathbf{a}_2) = \mathbf{a}_0 + \mathbf{a}_2 = \mathbf{a}_2 \neq \mathbf{a}_3$.

Which q is the best? As a measure of quality of a given granulation, it is natural to take the *worst-case* error probability, i.e., the error probability corresponding to the worst-case pair (i, j) (i.e., to the pair with the largest E_{ij}):

Definition 4.

- By an *error probability of a granulation*, we mean the value

$$E(q) \stackrel{\text{def}}{=} \max_{i,j} E_{ij}.$$

- The granulation with the smallest possible error probability is called *optimal*.

Theorem 3. *The granulation is optimal when q satisfies the equation*

$$q^3 - 5q^2 + 4q + 1 = 0,$$

i.e., when $q \approx 3.9$.

In short, here also $q \in [3, 4]$ is the optimal granulation scale.

4 Proofs

4.1 Proof of Theorem 1

1. Let us first prove that for every interval $I_i = [a_i - 3\sigma_i, a_i + 3\sigma_i]$ from an optimal granulation, $a_i = 6\sigma_i$.

Indeed, since all the intervals I_i must be reliably non-negative, we can conclude that $a_i - 6\sigma_i \geq 0$, hence $a_i \geq 6\sigma_i$. So, to complete this part of the proof, it is sufficient to show that we cannot have $a_i > 6\sigma_i$. We will prove this by showing that if $a_i > 6\sigma_i$, then the corresponding granulation can be improved.

Indeed, in this case, we can take $\sigma'_i = a_i/6 > \sigma_i$, and consider a wider interval $I'_i = [a_i - 3\sigma'_i, a_i + 3\sigma'_i] \supset I_i$. Due to our choice of σ'_i , this new interval is also reliably non-negative. Therefore, if we replace the interval I_i by I'_i , we still get a granulation, and $I_i \subset I'_i$, $I_i \neq I'_i$. Thus, the original granulation can be improved.

So, if the granulation is optimal (i.e., cannot be improved), we have $a_i = 6\sigma_i$.

2. Let us now prove that for every interval $I_i = [a_i^-, a_i^+]$ from an optimal granulation, $a_i^+ = 3a_i^-$.

Indeed, from Part 1 of this proof, we can conclude that for an arbitrary interval $I_i = [a_i^-, a_i^+] = [a_i - 3\sigma_i, a_i + 3\sigma_i]$ from the optimal granulation, we have $3\sigma_i = 0.5 \cdot a_i$, hence $a_i^- = a_i - 3\sigma_i = 0.5 \cdot a_i$ and $a_i^+ = a_i + 3\sigma_i = 1.5 \cdot a_i$. Thus, $a_i^+ = 3a_i^-$.

3. Let us now show that if two intervals from an optimal granulation intersect, then this intersection can only consist of a single point.

To prove this, we will show that if two intervals $I_i = [a_i^-, a_i^+]$ and $I_j = [a_j^-, a_j^+]$ have a more extensive intersection, then the granulation can be improved. Without losing generality, we can assume that $a_i^- \leq a_j^-$.

We already know that since both I_i and I_j are intervals from an optimal granulation, we have $a_i^+ = 3a_i^-$ and $a_j^+ = 3a_j^-$. Since $a_i^- \leq a_j^-$, we thus conclude that $a_i^+ = 3a_i^- \leq 3a_j^- = a_j^+$.

The fact that the intervals $I_i = [a_i^-, 3a_i^-]$ and $I_j = [a_j^-, 3a_j^-]$ have an intersection means that $a_j^- \leq 3a_i^-$; the fact that this intersection is not simply a single point means that $a_j^- < 3a_i^-$. In this case, $I_i \cup I_j = [a_i^-, 3a_j^-]$.

Let us show that we can improve the granulation if we replace I_i by itself $I'_i = I_i$ and I_j by $I'_j = [3a_i^-, 9a_i^-]$. Indeed, both new intervals are reliably non-negative, and the new union $I'_i \cup I'_j = [a_i^-, 9a_i^-]$ is a strict superset of the old one – because $a_j^- < 3a_i^-$ hence $3a_j^- < 9a_i^-$.

4. So, in an optimal granulation, every interval must be of the type $[a, 3a]$, these intervals must cover the entire real axis, and they cannot intersect in more than one point. Thus, right after each interval $[a_i, 3a_i]$, there should be the next interval $[a_{i+1}, 3a_{i+1}]$, so we should have $a_{i+1} = 3a_i$.

Thus, we get the description from the formulation of the theorem.

5. One can also easily prove that the granulation in which $I_i = [a_i, a_{i+1}]$ with $a_{i+1} = 3a_i$ cannot be improved and is thus optimal. The theorem is proven.

4.2 Proof of Theorem 2

In this proof, we will find the values of $P(\mathbf{a}_i + \mathbf{a}_j \in \mathbf{a}_k)$ for different i, j , and k , and then find, for each i and j , the value k for which the corresponding error $1 - P(\mathbf{a}_i + \mathbf{a}_j \in \mathbf{a}_k)$ is the smallest possible.

1. Let us start by computing the probabilities $P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_k)$ corresponding to $i = j$.

In this case, both A_i and A_j are uniformly distributed on the interval $[a_0 \cdot q^i, a_0 \cdot q^{i+1}]$. Possible pairs (A_i, A_j) are therefore uniformly distributed on the square $[a_0 \cdot q^i, a_0 \cdot q^{i+1}] \times [a_0 \cdot q^i, a_0 \cdot q^{i+1}]$.

- The smallest possible value of $A_i + A_j$ is $2a_0 \cdot q^i$ which (due to the assumption $q \geq 2$) belongs to the granule $[a_0 \cdot q^i, a_0 \cdot q^{i+1}]$.
- The largest possible value of $A_i + A_j$ is equal to $2a_0 \cdot q^{i+1}$. Due to $q \geq 2$, we have $a_0 \cdot q^{i+1} < 2a_0 \cdot q^{i+1} \leq a_0 \cdot q^{i+2}$, hence this value belongs to the next granule $[a_0 \cdot q^{i+1}, a_0 \cdot q^{i+2}]$.

So, the sum $A_i + A_i$ spreads over two different granules: \mathbf{a}_i and \mathbf{a}_{i+1} . Thus, of all possible probabilities $P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_k)$, only the values corresponding to $k = i$ and $k = i + 1$ are different from 0.

Since the pair (A_i, A_j) is uniformly distributed on the square, the probability $P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_i)$ is equal to the ratio between the area covered by the points for which the sum belongs to \mathbf{a}_i and the area of the entire square. This ratio does not change when we “rescale” the units by dividing both coordinates by a constant $a_0 \cdot q^i$. After this division, the square takes the form $[1, q] \times [1, q]$, and the pairs (x, y) in which we are interested are determined by the inequality $x + y \leq q$. Thus, $P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_i)$ is equal to the ratio S_i/S , where:

- $S = (q - 1)^2$ is the area of the square, and
- S_i is the area of the set of all points from this square for which $x + y \leq q$.

One can easily see that the set S_i is a right triangle with sides $[1, q - 1]$ of length $q - 2$. Thus, the area S_i of this triangle is equal to $(q - 2)^2/2$, and the desired probability is equal to

$$P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_i) = \frac{S_i}{S} = \frac{(q - 2)^2}{2(q - 1)^2}. \quad (1)$$

Since the sum $A_i + A_i$ can only belong to two possible granules \mathbf{a}_i and \mathbf{a}_{i+1} , we have $P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_i) + P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_{i+1}) = 1$, so $P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_{i+1}) = 1 - P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_i)$. Therefore:

- For $k = i$, the probability of error is equal to $1 - P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_i)$.
- For $k = i + 1$, the probability of error is equal to $1 - P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_{i+1}) = P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_i)$.

The triangle S_i is smaller than the half of the square, hence $P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_i) < 1/2$, and so $P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_{i+1}) = 1 - P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_i) > 1/2 > P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_i)$. So, for $i = j$, the error $1 - P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_k)$ is the smallest when $k = i + 1$. For this k , the error probability is equal to

$$E_{ii} = P(\mathbf{a}_i + \mathbf{a}_i \in \mathbf{a}_i) = \frac{(q-2)^2}{2(q-1)^2}. \quad (2)$$

2. Let us now find the values $P(\mathbf{a}_i + \mathbf{a}_j \in \mathbf{a}_k)$ for $i \neq j$.

Without losing generality, it is sufficient to consider only the case when $i < j$. Here:

- $A_i \in [a_0 \cdot q^i, a_0 \cdot q^{i+1}]$,
- $A_j \in [a_0 \cdot q^j, a_0 \cdot q^{j+1}]$, and
- the sum $A_i + A_j$ takes values from $a_0 \cdot (q^i + q^j)$ to $a_0 \cdot (q^{i+1} + q^{j+1})$.

The smallest possible value $a_0 \cdot (q^i + q^j)$ of the sum $A_i + A_j$ is greater than $a_0 \cdot q^j$ but is (due to $i \leq j$) smaller than $2a_0 \cdot q^j$ and hence (due to $q \geq 2$) smaller than $a_0 \cdot q^{j+1}$. Thus, this value belongs to the granule $\mathbf{a}_j = [a_0 \cdot q^j, a_0 \cdot q^{j+1}]$.

Similarly, the largest possible value $a_0 \cdot (q^{i+1} + q^{j+1})$ of the sum $A_i + A_j$ belongs to the next granule $\mathbf{a}_{j+1} = [a_0 \cdot q^{j+1}, a_0 \cdot q^{j+2}]$. Thus, out of all possible probabilities $P(\mathbf{a}_i + \mathbf{a}_j \in \mathbf{a}_k)$, only the values corresponding to $k = j$ and $k = j + 1$ are different from 0.

Similarly to the case $i = j$, the probability $P(\mathbf{a}_i + \mathbf{a}_j \in \mathbf{a}_j)$ is equal to the ratio of the two areas. Again, we can “rescale” both coordinates by dividing them by $a_0 \cdot q^i$, after which we get the following simplified geometric representation of the ratio $P(\mathbf{a}_i + \mathbf{a}_j \in \mathbf{a}_j)$ as S_j/S , where:

- S is the total area of the rectangle $[1, q] \times [q^d, q^{d+1}]$ (where $d \stackrel{\text{def}}{=} j - i$), i.e., the value $S = (q-1) \cdot (q^{d+1} - q^d) = q^d \cdot (q-1)^2$; and
- S_j is the total area of the set \mathcal{S}_j of all the points (x, y) from this rectangle for which $x + y \leq q^{d+1}$.

The set \mathcal{S}_j is bounded, from above, for $x = 1$, by the point $y = q^{d+1} - 1$, and for $x = q$, by the point $y = q^{d+1} - q$. Thus, this set \mathcal{S}_j consists of two subsets:

- a rectangle $[1, q] \times [q^d, q^{d+1} - q]$ of area $(q-1) \cdot (q^{d+1} - q^d - q)$ and
- a right triangle with sides $[1, q]$ and $[q^{d+1} - 1, q^{d+1} - q]$ of lengths equal to $q-1$ – whose area is equal to $(q-1)^2/2$.

Hence,

$$S_j = (q-1) \cdot (q^{d+1} - q^d - q) + \frac{(q-1)^2}{2} = q^q \cdot (q-1)^2 - (q-1) \cdot \frac{q+1}{2}. \quad (3)$$

3. We are now going to show that for $d \geq 2$, the area S_j is larger than a half of S ; then, similarly to the case $i = j$, we will be able to conclude that $\mathbf{a}_i + \mathbf{a}_j = \mathbf{a}_j$.

To prove this, it is sufficient to prove that the area of the complement \mathcal{S}_{j+1} to \mathcal{S}_j is not larger than one half of area of the the entire rectangle. The area S_{j+1} of this complement is equal to

$$S_{j+1} = (q-1) \cdot \frac{q+1}{2},$$

so this area does not exceed the half of $S = q^d \cdot (q-1)^2$ is and only if

$$(q-1) \cdot \frac{q+1}{2} \leq q^d \cdot \frac{(q-1)^2}{2}.$$

Dividing both sides of this inequality by $(q-1)/2$, we get a (simplified) equivalent inequality

$$q+1 \leq q^d \cdot (q-1). \quad (4)$$

Since $q \geq 2$ and $d \geq 2$, we have $1 \leq q^{d-1}$, and $q \leq q^{d-1}$, hence $1+q \leq 2q^{d-1} \leq q \cdot q^{d-1} = q^d$. We also have $q-1 \geq 1$, hence $q+1 \leq q^d \leq q^d \cdot (q-1)$, i.e., we have the desired inequality (4). Thus, for $|j-i| = d \geq 2$, we have $k = \max(i, j)$. For this case,

$$E_{ij} = \frac{S_{j+1}}{S} = \frac{q+1}{2q^d \cdot (q-1)}. \quad (5)$$

4. For $j-i = 1$, the inequality $S_{j+1} \leq S/2$ is equivalent to

$$(q-1) \cdot \frac{q+1}{2} \leq q \cdot \frac{(q-1)^2}{2},$$

i.e., to $q+1 \leq q^2 - q$, or to $q^2 - 2q - 1 \geq 0$. The corresponding equation $q^2 - 2q - 1 = 0$ has two roots $q = 1 \pm \sqrt{2}$, so for $q \geq 2$, the desired inequality is equivalent to $q \geq \sqrt{2} + 1$. So:

- For $q \geq \sqrt{2} + 1$, the sum $\mathbf{a}_i + \mathbf{a}_{i+1}$ is equal to \mathbf{a}_{i+1} , and the corresponding probability error is equal to

$$E_{i, i+1} = \frac{q+1}{2q \cdot (q-1)}. \quad (6)$$

- For $q < \sqrt{2} + 1$, the sum $\mathbf{a}_i + \mathbf{a}_{i+1}$ is equal to \mathbf{a}_{i+2} , and the corresponding probability error is equal to

$$E_{i,i+1} = 1 - \frac{q+1}{2q \cdot (q-1)}. \quad (7)$$

The theorem is proven.

4.3 Proof of Theorem 3

In the proof of Theorem 2, we have already found the error probabilities E_{ij} corresponding to different values i, j , and q . Let us use these probabilities to find the optimal q . Since the formulas for E_{ij} were different for $q \geq \sqrt{2} + 1$ and for $q < \sqrt{2} + 1$, we will consider these two cases separately.

Let us start with $q \geq \sqrt{2} + 1$. In this case, due to the formulas (2), (5), and (6), the error probability is equal to:

$$E(q) = \max \left(\frac{(q-2)^2}{2(q-1)^2}, \frac{q+1}{2q \cdot (q-1)}, \frac{q+1}{2q^2 \cdot (q-1)}, \dots, \frac{q+1}{2q^d \cdot (q-1)}, \dots \right). \quad (8)$$

Since $q \geq 1$, we have

$$\frac{q+1}{2q \cdot (q-1)} \geq \frac{q+1}{2q^2 \cdot (q-1)} \geq \dots \geq \frac{q+1}{2q^d \cdot (q-1)} \geq \dots,$$

and thus, to find the maximum, it is sufficient to consider only the first two terms:

$$E(q) = \max(E_1(q), E_2(q)), \quad (9a)$$

where

$$E_1(q) \stackrel{\text{def}}{=} \frac{(q-2)^2}{2(q-1)^2}; \quad (9b)$$

$$E_2(q) \stackrel{\text{def}}{=} \frac{q+1}{2q \cdot (q-1)}. \quad (9c)$$

Let us show that $E_1(q)$ is an increasing function of q , while $E_2(q)$ is decreasing.

- The expression $E_1(q)$ can be represented as

$$E_1(q) = \frac{1}{2} \cdot \left(1 - \frac{1}{q-1} \right)^2.$$

When q increases ($q \uparrow$), $q-1$ also increases ($q-1 \uparrow$), hence

$$\frac{1}{q-1} \downarrow, \quad 1 - \frac{1}{q-1} \uparrow, \quad E_1(q) = \frac{1}{2} \cdot \left(1 - \frac{1}{q-1} \right)^2 \uparrow.$$

- Similarly, if we divide both numerator and denominator of the expression $E_2(q)$ by q , we get the equivalent expression

$$E_2(q) = \frac{1 + 1/q}{2(q-1)}.$$

When $q \uparrow$, we have $1/q \downarrow$, hence $1 + 1/q \downarrow$. On the other hand, we have $2(q-1) \uparrow$, so $E_2(q)$ is the result of dividing a decreasing function by an increasing one – hence itself decreasing.

Since $E_1(q) \uparrow$ and $E_2(q) \downarrow$, we have $E_1(q) - E_2(q) \uparrow$.

- For $q = \sqrt{2} + 1$, we have

$$E_1(q) = \frac{3}{4} - \frac{1}{2} \cdot \sqrt{2}, \quad E_2(q) = \frac{3}{2} + \sqrt{2},$$

so $E_1(q) - E_2(q) < 0$.

- When $q \rightarrow \infty$, we have $E_1(q) \rightarrow 1/2$, while $E_2(q) \rightarrow 0$, hence for sufficiently large q , we get $E_1(q) - E_2(q) > 0$.

A continuous increasing function $E_1(q) - E_2(q)$ goes from a negative to a positive value, so it must attain the value 0 at some point q_0 . Let us show that on the set $[\sqrt{2} + 1, \infty)$, the minimum of $E(q)$ is attained for $q = q_0$. Indeed:

- For $q = q_0$, we have $E_1(q_0) = E_2(q_0)$ hence

$$E(q_0) = \max(E_1(q_0), E_2(q_0)) = E_1(q_0) = E_2(q_0).$$

- For $q < q_0$, we have $E_1(q) < E_2(q)$, hence

$$E(q) = \max(E_1(q), E_2(q)) = E_2(q).$$

Since the function $E_2(q)$ is decreasing and $q < q_0$, we have $E_2(q) > E_2(q_0)$, so $E(q) > E(q_0)$.

- Similarly, for $q > q_0$, we have $E_1(q) > E_2(q)$, hence

$$E(q) = \max(E_1(q), E_2(q)) = E_1(q).$$

Since the function $E_1(q)$ is increasing and $q > q_0$, we have $E_1(q) > E_1(q_0)$, so $E(q) > E(q_0)$.

Thus, among all values $q \in [\sqrt{2} + 1, \infty]$, the smallest possible value $E(q)$ is attained when $E_1(q) = E_2(q)$, i.e., when

$$\frac{(q-2)^2}{2(q-1)^2} = \frac{q+1}{2q \cdot (q-1)}.$$

Multiplying both sides of this equation by $2q \cdot (q - 1)^2$, we get the equivalent equation $q \cdot (q - 2)^2 = q^2 - 1$. Opening the parentheses and moving all the terms to one side, we get the desired equation from the formulation of Theorem 3.

For this value q , we have $E(q) \approx 0.21$. To complete the proof, we must show that for $q \in [1, \sqrt{2} + 1]$, the value of $E(q)$ is larger than 0.21. Indeed, for these q , we have

$$E(q) = \max \left(\frac{(q - 2)^2}{2(q - 1)^2}, 1 - \frac{q + 1}{2q \cdot (q - 1)}, \frac{q + 1}{2q^2 \cdot (q - 1)}, \dots, \frac{q + 1}{2q^d \cdot (q - 1)}, \dots \right),$$

hence

$$E(q) \geq 1 - \frac{q + 1}{2q \cdot (q - 1)}. \quad (10)$$

We already know that when $q \uparrow$, we have

$$\frac{q + 1}{2q \cdot (q - 1)} \downarrow, \text{ hence } 1 - \frac{q + 1}{2q \cdot (q - 1)} \uparrow$$

thence, for $q \geq 2$:

$$1 - \frac{q + 1}{2q \cdot (q - 1)} \geq 1 - \frac{2 + 1}{2 \cdot 2 \cdot (2 - 1)} = 0.25.$$

So, from (10), we conclude that $E(q) \geq 0.25 > 0.21$. The theorem is proven.

Conclusions

When people make crude estimates, they feel comfortable choosing between alternatives which differ by half-order of magnitude (e.g., were there 100, 300, or 1,000 people in the crowd), and less comfortable making a choice on a more detailed scale (like 100 or 110 or 120) or on a more spacious scale (like 100 or 1,000). We has shown that for two natural models of choosing granularity in commonsense estimates, in the optimal granularity, the next estimate is 3-4 times larger than the previous one. Thus, we explain the commonsense granularity.

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