# Is QA equivalent to SA ?

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11 June 2014

# Is QA equivalent to SA?

**Should be**

No!

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**But maybe**

Yes?

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## Quantize it!

$$|\psi\rangle = a|\mathrm{yes}\rangle + b|\mathrm{no}\rangle$$

# Classical dynamics to quantum Hamiltonian

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cf. Castelnovo, Chamon, Mudry, and Pujol, Ann. Phys. (2005)

# Example

- 1d ferromagnetic Ising model

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$$H_0(\sigma) = -J \sum_{j=1}^{N} \sigma_j \sigma_{j+1}$$

- $W$ of heat-bath dynamics (at fixed $T$) is equivalent to:

$$\hat{H} = -\frac{1}{2} \tanh 2K \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z - \frac{1}{2\cosh 2K} \sum_{j=1}^{N} \left( \cosh^2 K - \sinh^2 K \, \sigma_{j-1}^z \sigma_{j+1}^z \right) \sigma_j^x$$

# Some details for C to Q: Notation

- Matrix and vector of size $2^N (\times 2^N)$, $\sigma = \{\sigma_1, \sigma_2, \cdots, \sigma_N\}$

$$\left(\hat{H}_0\right)_{\sigma\sigma} = H_0(\sigma) \text{ (diagonal)}, \ \left(\hat{W}\right)_{\sigma\sigma'} = W_{\sigma\sigma'}, \ \left(\hat{P}(t)\right)_{\sigma} = P_\sigma(t)$$

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$$\frac{d\hat{P}(t)}{dt} = \hat{W}\hat{P}(t)$$

- (Right) eigenvector and eigenvalue

$$\hat{W}\hat{\psi}^{(R,n)} = -\lambda_n \hat{\psi}^{(R,n)},\ \lambda_0 = 0, -\lambda_1, \cdots$$

# General solution

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- Relaxation time

$$\tau = \frac{1}{\lambda_1}$$

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$$\Longrightarrow \hat{H}\hat{\phi}^{(n)} = \lambda_n \hat{\phi}^{(n)}$$

# Matrix elements of $\hat{H}$

## Off-diagonal

$$H_{\sigma\sigma'} = -e^{\frac{1}{2}\beta H_0(\sigma)} W_{\sigma\sigma'} e^{-\frac{1}{2}\beta H_0(\sigma')} = -w_{\sigma\sigma'} \ (< 0)$$

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## Combined: operator representation

$$\hat{H} = \sum_{\sigma} \sum_{\sigma' \in \mathcal{N}(\sigma)} \left( w_{\sigma\sigma'} e^{-\frac{1}{2}\beta(H_0(\sigma') - H_0(\sigma))} |\sigma\rangle\langle\sigma| - w_{\sigma\sigma'} |\sigma'\rangle\langle\sigma| \right)$$

# Locality + single-spin flip

- Assume $H_0(\sigma)$ is local.

$$H_0(\sigma) = \sum_j H_j, \quad \left(H_j = -h_j\sigma_j - \sigma_j \sum_{k\in\mathcal{N}(j)} J_{jk}\sigma_k - \cdots\right)$$

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- Assume $\sigma \to \sigma'$: $\sigma_j \to -\sigma_j$ (single-spin flip)

$$H_0(\sigma) - H_0(\sigma') = H_j - (-H_j) = 2H_j \quad \text{(local)}$$

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$$= \sum_j w(\sigma_j^z \to -\sigma_j^z)\big(e^{\beta H_j}\mathbb{I} - \sigma_j^x\big)$$

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Local Hamiltonian!

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- Heat-bath dynamics

$$\hat{H} = (\text{const}) - \frac{1}{2}\tanh 2K \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z$$

$$- \frac{1}{2\cosh 2K} \sum_{j=1}^{N} \left( \cosh^2 K - \sinh^2 K \, \sigma_{j-1}^z \sigma_{j+1}^z \right) \sigma_j^x$$

# Adaptive change of local transverse fields

- Transverse-field term

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- Weak/strong field for desirable/undesirable configuration
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- cf. Uniform transverse field → quantum phase transition

# Homework

Show that the quantum Hamiltonian for the classical dynamics of

$$\hat{H}_0 = -h\sigma_z$$

is

## Heat-bath dynamics

$$\hat{H}\left(= -e^{\frac{1}{2}\beta\hat{H}_0}\hat{W}e^{-\frac{1}{2}\beta\hat{H}_0}\right) = \frac{1}{2} - \frac{1}{2}\tanh\beta h\,\sigma_z - \frac{1}{2\cosh\beta h}\sigma_x$$

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Rewrite the master equation in terms of $\hat{\phi}(t)$ and $\hat{H}(t)$

$$\frac{d\hat{\phi}(t)}{dt} = -\hat{H}(t)\hat{\phi}(t) + \frac{1}{2}\dot{\beta}(t)\hat{H}_0\hat{\phi}(t)$$

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## $t \to it$: Schrödinger equation

$$i\frac{d\hat{\phi}(t)}{dt} = \left(\hat{H}(t) - \frac{1}{2}\dot{\beta}(t)\hat{H}_0\right)\hat{\phi}(t)$$

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  cf. Classical to quantum: $\hat{\phi}^{(0)} = e^{-\frac{1}{2}\beta\hat{H}_0}/\sqrt{Z}$

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- Transition matrix: $\hat{W} := -e^{-\frac{1}{2}\hat{H}_0}\hat{H}e^{\frac{1}{2}\hat{H}_0}$

## Summary

- Equivalence: Eigenvalue spectrum (fixed $T$), Time-dependent $T(t)$

$$\hat{W}, -\hat{H} : \lambda_0 = 0 > -\lambda_1 > -\lambda_2 > \cdots$$

$$(\text{Relaxation time})^{-1} = \text{Energy gap}$$

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|---|---|
| classical $\to$ quantum | short |
| quantum $\to$ classical | long |

- Thanks to Junichi Tsuda

# Transition matrix

## Properties to be satisfied

$$W_{\sigma\sigma'} \geq 0 \quad (\sigma \neq \sigma'), \quad W_{\sigma\sigma} = -\sum_{\sigma'(\neq\sigma)} W_{\sigma'\sigma} \text{ (diagonal)}$$

$$W_{\sigma\sigma'} P_{\sigma'}^{(0)} = W_{\sigma\sigma'} P_{\sigma}^{(0)} \quad \text{(detailed balance)}$$

$$W_{\sigma\sigma'} = w_{\sigma\sigma'} e^{-\frac{1}{2}\beta\left(H_0(\sigma)-H_0(\sigma')\right)} \quad (W_{\sigma\sigma'} \neq W_{\sigma'\sigma})$$

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- Example: Heat-bath method

$$w_{\sigma\sigma'} = \frac{1}{e^{-\frac{1}{2}\beta(H_0(\sigma')-H_0(\sigma))} + e^{\frac{1}{2}\beta(H_0(\sigma')-H_0(\sigma))}}$$

$$\Longleftrightarrow W_{\sigma\sigma'} = \frac{e^{-\beta H_0(\sigma)}}{e^{-\beta H_0(\sigma')} + e^{-\beta H_0(\sigma)}}$$

# Equilibrium vs ground state

- Wave functions

$$\hat{W} : \hat{\psi}^{(R,0)} = \frac{1}{\sqrt{Z}} e^{-\beta \hat{H}_0} \sum_{\sigma} |\sigma\rangle = \frac{1}{\sqrt{Z}} \sum_{\sigma} e^{-\beta H_0(\sigma)} |\sigma\rangle$$

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- Expectation value of $\hat{Q}$ (diagonal in $\sigma$)

$$\langle \hat{\phi}^{(0)} | \hat{Q} | \hat{\phi}^{(0)} \rangle = \frac{1}{Z} \sum_{\sigma} Q(\sigma) e^{-\beta H_0(\sigma)} = \langle Q \rangle_T$$