Spin-glass error-correcting codes with quantum decoding

Ching-Yi Lai and Yicong Zheng

Electrical Engineering Department
University of Southern California

April 26, 2012

Outline

Linear Block Codes

- Definition: An $[n, k, d]$ binary linear block code C is a k-dimensional subspace of all binary n-tuples $\mathbb{Z}_2^n$ such that any two n-tuples in this code differ in at least d bits.
- d is called the *minimum distance* of the linear code.
- We can choose a basis of the codespace: $\{g_1, g_2, \cdots, g_k\}$.
- The matrix

$$G = \begin{bmatrix} -g_1- \\ -g_2- \\ \vdots \\ -g_k- \end{bmatrix}$$

  is called the *generator matrix* of the code C. C is the rowspace of G.
- A k-bit information $u = a_1 \cdots a_k$ is encoded into a codeword $x \in C$ by

$$x = uG = a_1 g_1 + a_2 g_2 + \cdots + a_k g_k.$$

Duality of Linear Codes

- The *orthogonal space* of C, under the usual inner product of binary vectors, is an $[n, n-k]$ code space, and is denoted by $C^{\perp}$.
- Denote the generator matrix of the orthogonal code $C^{\perp}$ by H.
- A legal codeword $x \in C$ has to satisfy

$$Hx^T = 0.$$

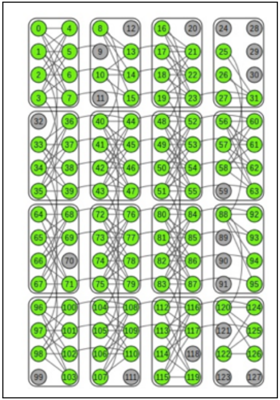
  Then we have

$$HG^T = O.$$

- H is called the *parity-check matrix* of the code C.

The D-Wave Chip

- The D-Wave chip corresponds to a specific 2-dimensional Ising model of 128 qubits.
- Only 108 of the 128 qubits are working qubits.
- 255 pairs of qubits are coupled.



Spin-glass Codes

- Let $i_1, \cdots, i_{108}$ be the numbers of the working qubits and we denote these working qubits by $s_{i_1}, \cdots, s_{i_{108}}$ and assume they carry classical messages:

$$s_{i_m} \in \{\pm 1\}.$$

- Let E denote the edge set

$$\{(a, b) : s_a \text{ and } s_b \text{ are coupled}\},$$

and $|E| = 255$.

- If two qubits s_a and s_b are coupled, the parity of s_a and s_b is encoded in the "*edge*" $J_{a,b}$:

$$J_{a,b} = s_a s_b.$$

- Map $+1, -1$ to $0, 1$, respectively. This is equivalent to a linear block code.

Symmetry of Spin-glass Codes

- The spin-glass code has a 108×255 generator matrix G' such that each column is of weight 2.
- Suppose $x = s_{i_1} \cdots s_{i_{108}}$ is a message string. Let $\bar{x} = (-s_{i_1}) \cdots (-s_{i_{108}})$. It can be observed that x and $\bar{x}$ are encoded to the same codeword, since

$$J_{a,b} = s_a s_b = (-s_a)(-s_b)$$

for any $(a, b) \in E$.
- The spin-glass code is equivalent to a $[255, 107]$ low-density generator matrix (LDGM) code.

Cyclic Codes

- A code C is cyclic if it is linear and any cyclic shift of a codeword is also a codeword.
- If $(c_0, c_1, \cdots, c_{n-1}) \in C$, $(c_1, c_2, \cdots, c_{n-2}) \in C$.
- Let $c(x) = c_0 + c_1 x + \cdots c_{n-1} x^{n-1}$ denote a codeword $(c_0, c_1, \cdots, c_{n-1}) \in F^n$, where $F = GF(q)$.
- A cyclic code C is an *ideal* of $R_n = F[x]/(x^n - 1)$:
  - If $c(x) \in C$, then $r(x)c(x) \in C$ for any $r(x) \in R_n$.
- $C = \langle g(x) \rangle$ for some generator polynomial $g(x)$.

Bose-Chaudhuri-Hocquenghem (BCH) Codes

- A cyclic code of length n over $GF(q)$ is called a BCH code with designed distance δ if one of its generator polynomial is of the form
$$g(x) = \prod_{i \in K} (x - \alpha^i).$$
- α is a primitive p-th root of unity
- $K = C_b \cup \ldots \cup C_{b+\delta-2}$ is the defining set of the code
- $C_s = \{q^i s \mod n \mid i \in K\}$ is the cyclotomic coset mod n over $GF(q)$ containing s.
- If $n = q^m - 1$ for some m, it is called primitive. If $b = 1$, it is called a narrow-sense BCH code.

BCH Codes

- A comparable binary BCH code of $n = 255$ and $k = 107$ is a $[255, 107, 45]$ primitive binary BCH code.
- Decoder: Berlekamp-Massey decoding algorithm (implemented in MATLAB)
- The $[255, 107, 45]$ can be cyclicly generated by the following vector:

  100000000000000000000000000000000000000
  000000000000000000000000000000000000000
  000000000000000000000000000001000010010
  011011000001011110011111010111111000000
  100011110011101011100100000001101111000
  010101001100111010100110110000100101100
  101111111010000100100,

  which is found by MAGMA.

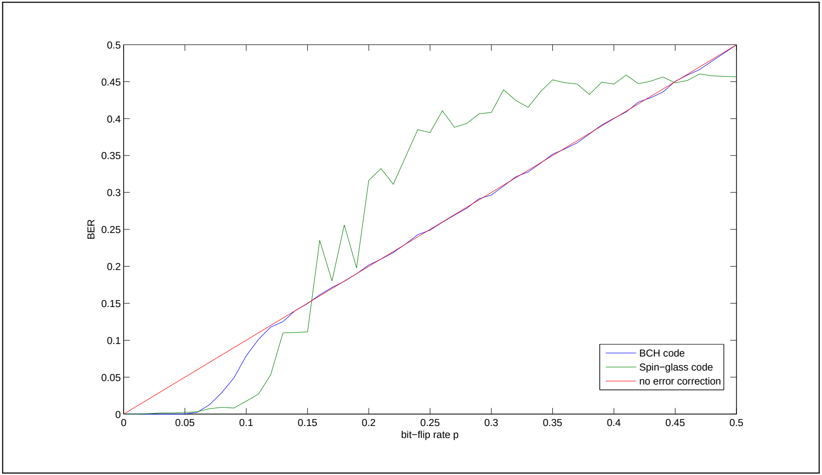
Bit Error Rate (BER) over Binary Symmetric Channel (BSC)

- Assume the noise channel is the binary symmetric channel (BSC) with bit-flip rate p.
- The channel capacity of a binary symmetric channel is

$$C = 1 - H(p),$$

  where $H(p) = -p \log p - (1-p)\log(1-p)$.
- Code rate $R = 107/255$.
- WLOS, we can assume that the all-zero vector is sent every time.
- $$\text{BER} = \frac{\text{sum of the weight of information bits}}{\text{total number of information bits}}.$$

Classical analysis



- For $R = 107/255$, it is better doing error-correcting codes than without error correction when $p < 0.1385$.
- BCH code is better for $0 < p < 0.06$.

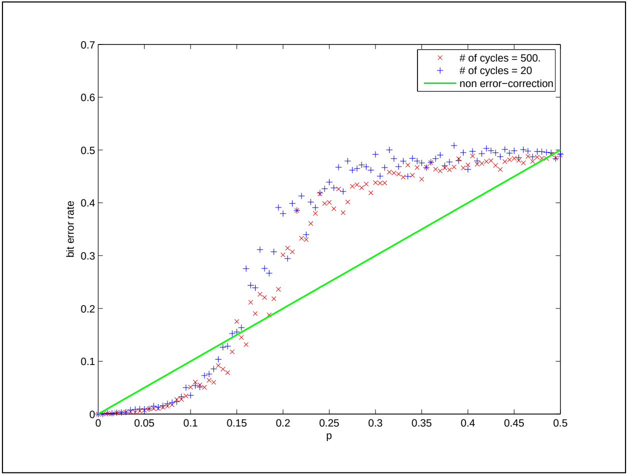
Simulation Environment

- classical local solver
- num_programming_cycle=number of times setting coupling coefficients and magnetization field = 100
- num_reads = number of quantum annealing for each = cycle 5
- number of noise sample = 60

Highly Degenerated Ground Space

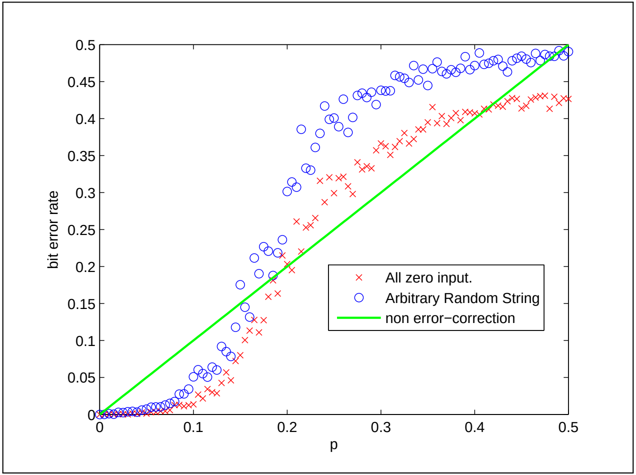
- Ground Space is in general highly degenerated
- We need to find all the ground state, in order to find the one with minimum weight, otherwise it introduces additional decoding error
- In classical case, we should run enough cycles to ensure all ground states has been found
- However, in quantum annealing algorithm, it doesn't guarantee running enough cycles can find all ground states

Simulation with Small number of cycles



- Input arbitrary random string

Input all zeroes string and arbitrary random string



- Reason?

Modified Spin Glass Codes

- What if modified the topology of the chip?
- Eliminate edges on the graph just reduces the number of N, thus makes code worse
- Eliminate specified qubits and edges connected to them. (Reduce N and K simultaneously.)
- Qubits with small number of edges may affect the performance more than the others.

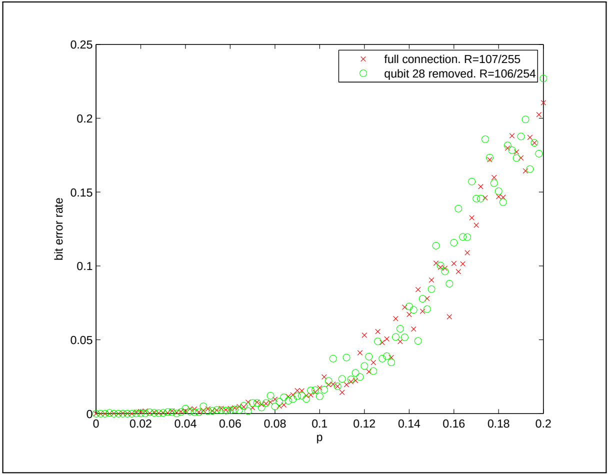
Shortened Codes

- This spin-glass code has a codeword of weight 1. This is because the qubit 28 is only coupled to one qubit (32)
- The qubit has two edges on the chip:
  qubit 26, to qubit 32 (3 edges) and qubit 58 (5 edges)
  qubit 27, to qubit 32 (3 edges) and qubit 59 (5 edges)
  qubit 93, to qubit 85 (6 edges) and qubit 89 (6 edges)
  qubit 94, to qubit 86 (6 edges) and qubit 89 (6 edges)
  qubit 95, to qubit 87 (6 edges) and qubit 89 (6 edges)
  qubit 96, to qubit 88 (6 edges) and qubit 89 (6 edges)
  qubit 127, to qubit 121 (4 edges) and qubit 123 (3 edges)

Shortened Codes

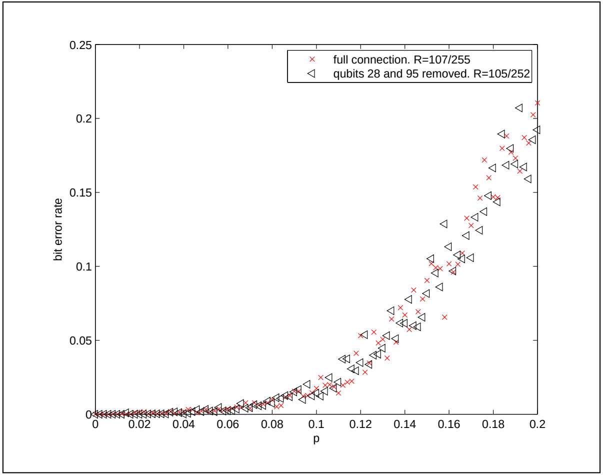
- Remove qubit 28
- qubits 28 and 93
  qubits 28 and 94
  qubits 28 and 95
  qubits 28 and 96
  qubits 28, 93 and 94
  qubits 28, 93 and 95
  qubits 28, 93 and 96
  qubits 28, 94 and 95
  qubits 28, 94 and 96
  qubits 28, 95 and 96
  qubits 28, 89, 93, 94, 95, and 96

Performance



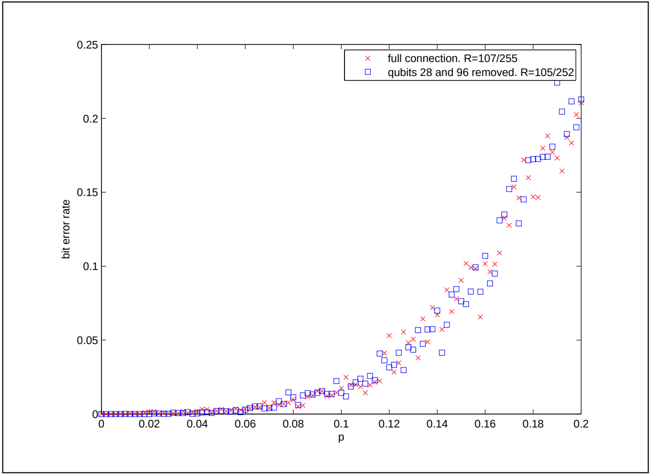
- Remove qubit 28

Performance (cont')



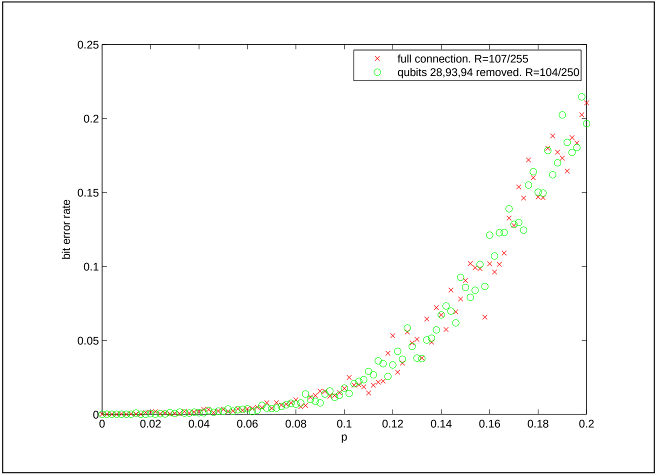
- Remove qubit 28 and 95

Performance (cont')



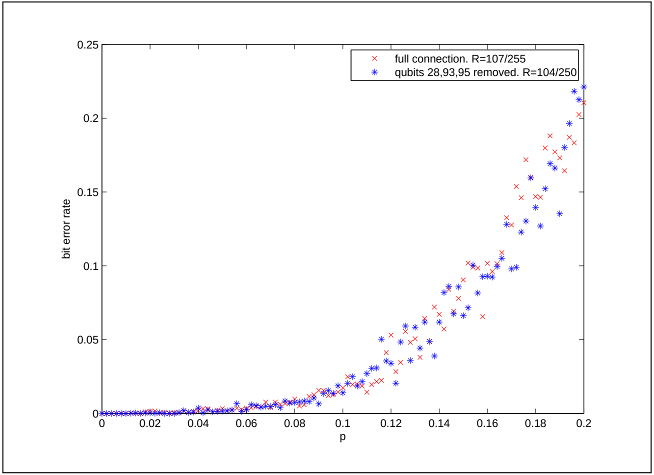
- Remove qubit 28 and 96

Performance (cont')



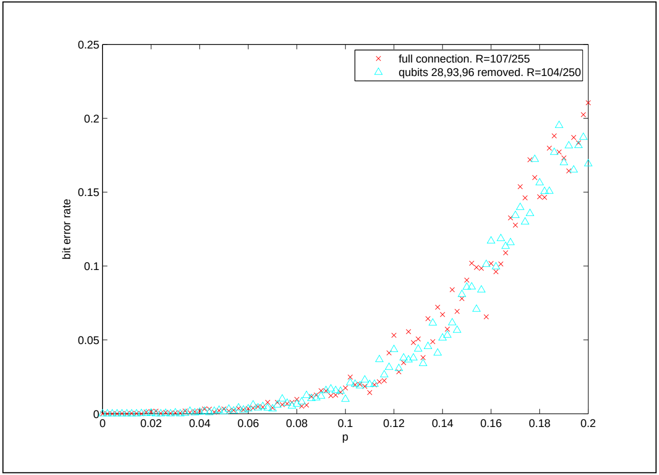
- Remove qubit 28, 93, and 94

Performance (cont')



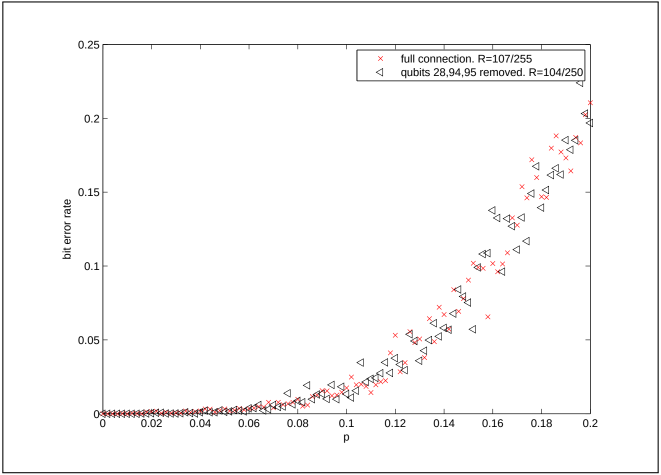
- Remove qubit 28, 93, and 95

Performance (cont')



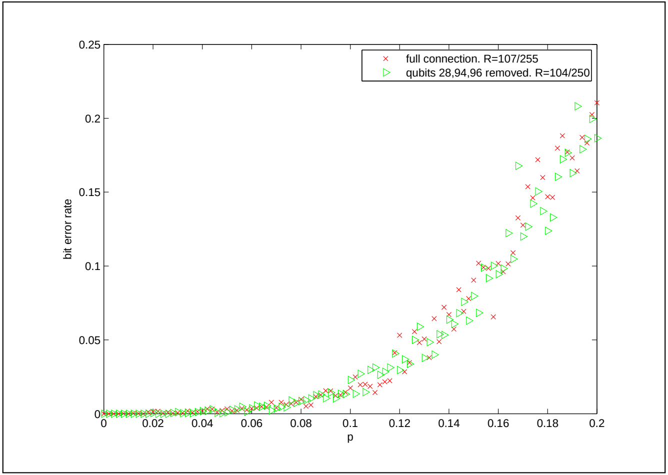
- Remove qubit 28, 93, and 96

Performance (cont')



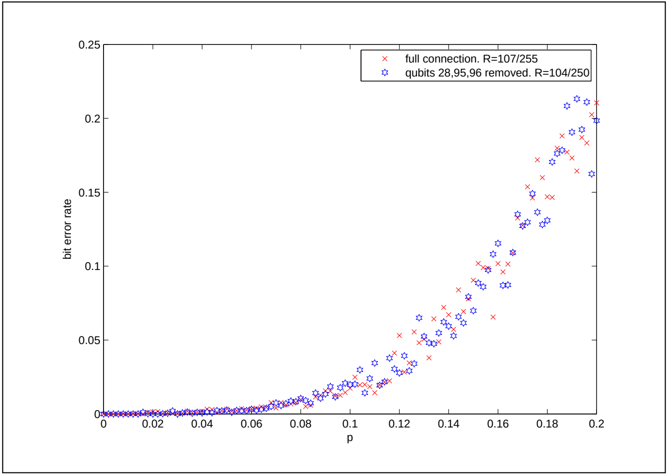
- Remove qubit 28, 94, and 95

Performance (cont')



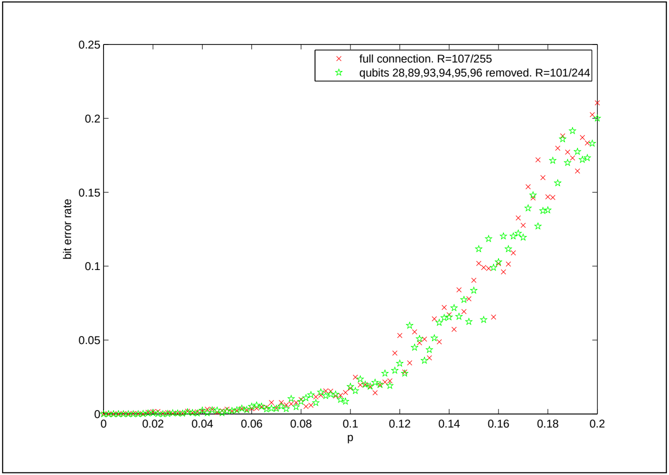
- Remove qubit 28, 94, and 96

Performance (cont')



- Remove qubit 28, 95, and 96

Performance (cont')



- Remove qubit 28, 89, 93, 94, 95, and 96

Future Work

- Check the performance of D-Wave Decoder
- Whether the performance of quantum decoder is better than the classical decoder of this spin glass code?
- The spin-glass code is equivalent to a LDGM code with a generator matrix of all weight-2 columns. How to couple pairs of qubits such that the equivalent LDGM code is good?
- Is it possible to find a shortened code with good error-correcting ability from this spin-glass code?

Reference

- F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*. Amsterdam, The Netherlands: North-Holland, 1977.
- S. Lin and J. Daniel J. Costello, *Error Control Coding*. New Jersey: Pearson Prentice Hall, 2004.